BOOK REVIEWS

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Generalized cohomology, by Akira Kono and Dai Tamaki, Translations of Mathematical Monographs, vol. 230, American Mathematical Society, Providence, RI, 2006, xi+254 pp., US$49.00, ISBN 978-0-8218-3514-2

The time from the late forties through the late sixties was a revolutionary era for algebraic topology. New theories having formal properties similar to those of the classical theory were constructed by means of novel pieces of algebra together with deep and unexpected results in homotopy theory. The new theories from those times, known as K-theory and cobordism theory, have found a wide range of applications in other subjects as well as extending the scope of algebraic topology itself. The term generalized homology/cohomology came into use to provide a common name for theories having the same formal properties as the traditional theory, with the single exception that they produce different values on spheres.

A half-century later one could expect most of these developments to have been treated in the combination of comprehensive texts, monographs, and introductory books similar to what is found for other subjects emerging from the same earlier era. There are some very good contributions in each of these categories, but their number is small and there is plenty of room for more. I like to think that the relative paucity of treatments is due to the extraordinarily high level of the exposition in many of the basic research papers. However, learning this material can be difficult without an expert at hand to explain things. The aim of the present book is essentially to provide a reasonably comprehensive guide to the newer results that can be read with only the standard background in algebraic topology. Eventually in this review, I will discuss the book in some detail, but now let me say that at the very least, the book provides students and other newcomers with the language needed to converse with an expert.

The book is a successful guide because it provides a smooth path from basics into the deeper parts of complex K-theory and complex cobordism theory (and by analogy into related theories). It supplies details for some crucial theorems (e.g. Bott periodicity for the infinite unitary group and the homotopy groups of MU, the Thom spectrum for complex cobordism theory) and directs the reader to excellent treatments of quoted material. However, as travel on superhighways, bypassing tricky curves and dangerous intersections of earlier roads, may leave one curious about the missing scenery, the smooth condensed treatment in the present book

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may leave one to wonder how the revolutions in the subject took place. Thus I want to devote a portion of this review to describing how we got from ordinary algebraic topology as treated in almost any first course to the present subject.

At first glance, “generalized cohomology” does not bear much resemblance to ordinary cohomology. The formal properties are there, but the means to achieve them do not use any of the constructions based on chain complexes. Moreover, familiar words like “orientability” appear but with strange new meanings. Then there are things that seem to tumble from the sky, notably, formal groups. The story of how these changes occurred includes radical shifts in viewpoint, with surprising theorems having far-reaching consequences. Great names march across the scene, but their work, while fundamental, would not have resulted in the present subject without the support of many workers of great abilities and imagination. The bibliography, with approximately 150 items, mostly research papers, reads like a cast list for a major production that might be called “Stable Algebraic Topology” (and these words have been used, without theatrical overtones, to identify the subject within all of topology).

The first sign of a revolution was a change in viewpoint regarding ordinary cohomology. But before we come to that let’s start with ordinary homology. Some casual conversations with my colleagues suggested that if one were not a topologist and were not using some of its results, then algebraic topology is only the name of a course taken long ago. Even those with some feeling for the subject wanted to think about it in geometric terms. They could appreciate a theory that produces fixed point theorems, invariance of domain, and several duality relationships by means of “functorial” properties and at the same time regret the loss of contact with geometry in the formal development.

Now the new theories from the revolutionary era are directly rooted in geometry, and this feature accounts, in part, for their success. K-theory starts with vector bundles. Cobordism revives the intuitive idea of bounding harbored in the original work of Poincaré but later sacrificed in order to construct a viable theory based on simplicial complexes. However, while more geometric than classical theory, the new theories must incorporate the same formal structure in order to achieve their results. So the story of their creation may provide a new perspective on the contact between geometry and the formal structure of homology theories. I propose to tell this story in three episodes: the representation theorem for ordinary cohomology, from Eilenberg-Mac Lane spaces to generalized cohomology, and from orientability to formal groups. Darting in and out is the marvelously inventive work of René Thom.

First episode. Ordinary cohomology came on the scene in the thirties. At first it served as an exceptionally convenient means to discuss new constructions and reinterpret older parts of topology. An excellent example, and precursor to the main event in this episode, is the Hopf-Whitney theorem. This theorem asserts a one-to-one correspondence between the n-th integral cohomology group of an n-dimensional space and the set of homotopy classes of maps from that space to the n-sphere. On one side we have a computable algebraic object and on the other side, an important geometric object.

Our story continues with the observation by Hurewicz that the homology groups of a space having a contractible covering space are determined, somehow, by the fundamental group. Through work of Hopf, Eckmann, Eilenberg and Mac Lane, the process was revealed and the subject of group homology emerged. Perhaps because they could, Eilenberg and Mac Lane introduced spaces $K(A,n)$ that generalized the original case, $n = 1$. In particular, the new spaces have a single non-zero homotopy group A (necessarily abelian) in degree n. The theorem for the eponymous spaces is that the homology of $K(A,n)$ is determined by the group A and the integer n. Moreover, they produced the bar construction for calculation. It was hoped that important information would emerge.

The realization of that hope was achieved by Serre with the observation that cohomology is represented by maps to Eilenberg-Mac Lane spaces and with the application of powerful new methods originating with Leray for determining their cohomology when A is cyclic of order 2. The Cartan seminar supplied results for other groups, including all finitely generated abelian groups and the additive group of rational numbers. This work vastly extended the scope of classical algebraic topology and paved the way for our next episode.

Second episode. With the spaces $K(A,n)$ in hand it is possible to develop classical algebraic topology entirely in terms of elementary homotopy theory, and without the aid of chain complexes, where the n-th cohomology group with coefficients in the abelian group A is defined by the representation theorem. The organizing idea is the notion of a "spectrum". This is nothing more than a collection of spaces $\{E_n\}$ indexed by all integers (this is important) together with maps from the suspension of the n-th space to the $(n+1)$-st space for each integer n. In the classical case E_n is $K(A,n)$ for $n > 0$, A as a discrete space for $n = 0$, and a point for the negative integers. This development might have been of only pedagogical interest if it were not for the arrival (as if in the overnight mail) of two new theories having the formal properties of classical cohomology with the single exception of different values when applied to spheres.

The new theories arise because of new ways to attach an abelian group to a space and novel theorems for identifying the representing spectra. For example, Atiyah and Hirzebruch apply a general construction of Grothendieck to the isomorphism classes of complex vector bundles over a space X. The resulting abelian group is denoted $K(X)$. Classical theorems about the representation of vector bundles by maps to Grassmannians lead to a space BU (the classifying space for the infinite unitary group) that serves to represent K-theory. The remarkable Bott periodicity theorem provides the spectrum for complex K-theory. Significant features of BU, especially characteristic classes, are now available for novel uses. Just as spectacular was the work of Thom that led to cobordism theory.

Cobordism theory is based on a piece of geometry that was introduced by Pontrjagin in the late thirties to study homotopy groups of spheres, then revisited and redirected by Thom in the early fifties. I will try to describe Pontrjagin's construction in a special case, because it provides a way to visualize the third homotopy group of the 2-sphere by playing with a belt. We look at ways to embed solid tori in the 3-sphere (think R^3). A solid torus is the product of a circle C and a 2-disc D. Embed this in space so that the circle describes a smooth knot and the image looks like a tube with the knot in its center but with internal twistings arising from the way the solid torus is mapped. These things come from smooth maps of the 3-sphere to the 2-sphere by considering the inverse image of a nice round neighborhood of a regular point on the 2-sphere. They are called framed embeddings.

Conversely, starting with a framed embedding, we can produce a map from the 3-sphere to the 2-sphere. This is done by projecting the image of the product $C \times D$ in the 3-sphere onto the image of D, collapsing the boundary of the image of D to a point P to produce the 2-sphere, and mapping the complement of the image of $C \times D$ to the point P. The remarkable fact is that the homotopy class of mappings constructed in this manner does not depend on how the circle C is embedded; knotting is irrelevant here. Only the number of twists in the framed embedding of the unknot matters and this number is a homotopy invariant that accounts for the entire homotopy group, with the Hopf map having invariant one. Thom makes an ingenious combination of this construction with the universal bundles for real rotation groups to produce the Thom spectrum. An analogous construction for unitary groups produces the Thom spectrum MU for complex cobordism theory. The homotopy groups of MU, also known as the coefficient ring for the theory, have an astonishing property, and we pass to the final episode in our story.

Third episode. Formal groups arrive in algebraic topology with the appearance of Quillen's work. There is an element of magic about all this, but after the fact we may trace a line of thought beginning with the familiar notion of orientability.

The intuitive idea of orientability as a coherent choice of orientations on coordinate charts on a manifold is well captured by ordinary integral homology. Fix the dimension of a closed manifold M at n. The choice of orientation on a single coordinate chart corresponds to a choice of a generator for the n-th integral homology group of the coordinate pair $(R^n, R^n - 0)$; this group is the integers. An orientation on M, if it exists, is an element called the fundamental class of M in the n-th integral homology group of M that determines coherent local choices from the maps induced on homology by two inclusions. The coordinate chart describes an inclusion of the coordinate pair to the pair $(M, M - pt)$ that is an excision. For each point in M we also have the identity map from M to the pair $(M, M - pt)$.

There is a similar notion of orientability for real vector bundles E over a space X. Let E_0 denote the complement of the zero-section and consider the inclusion of each fiber pair $(R^n, R^n - 0)$ in the pair (E, E_0). An orientation on the vector bundle is a cohomology class U in the n-th integral cohomology of the pair (E, E_0) that maps (contravariantly) to a generator of the n-th integral cohomology of the fiber pair; the group is the integers. Such a class is known as the Thom class. This idea of orientability extends the previous idea, because an orientable manifold has an orientable tangent bundle. Thom classes have several remarkable properties. One of these is expressed as the Thom isomorphism theorem, and this theorem is the next step on the way to formal groups.

The Thom isomorphism theorem for oriented bundles, originally formulated for classical cohomology, can be proved for generalized theories whenever there is a Thom class as above, with the generalized theory replacing the classical theory. Now we come to a slight shift of viewpoint with large ramifications. We say, following Frank Adams, that a theory is orientable provided any complex line bundle has a Thom class for that theory.

Now formal groups enter the picture, because for an orientable theory, the H-space structure on infinite dimensional complex projective space determines a formal group over the coefficient ring for the theory (its values on a point). Quillen's remarkable theorem is that the coefficient ring for complex cobordism theory is Lazard's universal formal group. This work unleashed a vast arsenal of algebraic tools new to topology, and we end our story to turn to the book that is to be reviewed.

The book. Algebraic topologists will probably know what to expect, so for the most part, I will ignore them in favor of an attempt to convey what a "non-topologist" could glean from dipping into the material and skipping technical parts. This audience can skip the first chapter as long as words like "space" and "homotopy" have meaning. The second chapter is a careful account of the formalism, and an expert will recognize how it is crafted for present uses. As it concerns formalities, any mathematician can read it. In addition, its opening pages give a clear description of the constructions that produce the spectrum for complex K-theory and the Thom spectrum for the rotation group.

The third chapter presents significant new content. The focus is on complex oriented theories, and someone for whom the subject is new can understand the statements and follow the spirit of the proofs. Among other things, the way characteristic classes enter this subject is carefully spelled out.

For "non-topologists" the fourth chapter will divide into two parts. The first four sections develop complex K-theory from commonly understood facts about vector bundles. It includes a treatment of the Bott periodicity theorem for the classifying space BU and a development of the Adams operations. The proof of the Hopf invariant one theorem using this structure is an excellent illustration of the impact of the theory of characteristic classes through K-theory. The rest of this chapter sketches ideas concerned with fiber homotopy type and the Adams conjecture. I liked the summary of representation theory for this work.

The fifth chapter is an excellent account of spectral sequences. An expert recognizes the need for this machinery, a newcomer can ignore it.

The sixth chapter is about complex cobordism. In earlier parts of the book some of the ideas have been introduced. The book does not pursue all the ramifications of Thom's revolutionary work but is of help towards learning them in other sources. The focus here is on the fact that complex cobordism is a complex orientable theory and on the calculation of the coefficient ring. At this point, formal groups are present and Quillen's theorem is proved with some quotes from available texts. Some further remarks concerning the use of the Landweber exact functor theorem to produce new theories are made. With this we are back where we started, with theories somehow separated from their geometric content.

The sentence above was fun to write but totally unfair to the subject. The stable homotopy ring has important information for mathematics. It also has abhorrent properties, it is nilpotent, and most finitely generated ideals are not finitely presented. Amazingly, formal groups seem to provide real insight to its structure by way of a construction known as the chromatic filtration. I can't explain this and it is not treated in this book, but the book is a good place for the novice to start.

There are three appendices, each with helpful accounts of technical material for the subject. Here I just report their titles: simplicial techniques, limits, and spectrum.

This book has been carefully prepared. In particular, the diagrams, which are a necessary part of the exposition, are well situated on their pages. There are very few typos, but I note one that might flummox a novice. On page 54, in the second paragraph, the symbol for the canonical line bundle should be replaced with the symbol for the universal bundle.

In conclusion, I recommend the book to those with an interest in this material and never mind prior background. The book has a lot of interesting information and paves the way to comprehensive texts and the literature.

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