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Harmonic measure: Geometric and analytic points of view, by Luca Capogna, Carlos E. Kenig, and Loredana Lanzani, University Lecture Series, vol. 35, Amer. Math. Soc., Providence, RI, 2005, x+155 pp., US$28.00, ISBN 978-0-8218-3920-1

This short book is an exposition of Carlos Kenig's 2000 Arkansas Spring Lecture Series. It explains three deep papers by Kenig and Tatiana Toro on the relation between the harmonic measure for a domain $\Omega \subset \mathbb{R}^{n+1}, n \geq 2$, and the smoothness of $\partial\Omega$. For simplicity let us assume $\Omega \subset \mathbb{R}^{n+1}$ is a bounded connected open set such that $\partial\Omega$ is connected and such that every $Q \in \partial\Omega$ is regular for the Dirichlet problem:

$$\Delta u = 0 \text{ on } \Omega;\ u|_{\partial\Omega} = f \in C(\partial\Omega). \tag{1}$$

Then for each $x \in \Omega$ there is a probability measure ω^x on $\partial\Omega$ called *harmonic measure* for x such that the solution of (1) is

$$u(x) = \int_{\partial\Omega} f(Q) d\omega^x(Q).$$

By Harnack's theorem the harmonic measures for different x_1 and $x_2 \in \Omega$ are mutually absolutely continuous with bounded densities, and we only study $\omega = \omega^{x_*}$ for one fixed $x_* \in \Omega$.

We set the stage by stating two older theorems for smooth domains and recalling the much easier situation of $\mathbb{R}^2$. Next we will describe the three types of domains treated in the book [B]: nontangentially accessible domains (NTA), Reifenberg flat domains, and chord-arc domains. Along the way we will explain the Geometry $\Longrightarrow$ Potential Theory theorems from Chapter 6 and the more difficult Potential Theory $\Longrightarrow$ Geometry theorems of Chapter 7. Finally, we will survey some more recent results and discuss the book itself.

1. Very Smooth Domains. When $\partial\Omega$ is locally the graph of a $C^{1+\alpha}$ function, it follows from Green's theorem that $\omega \ll \sigma$ where $\sigma = H^n|\partial\Omega$ is surface measure, and when $\omega \ll \sigma$ the Radon-Nikodym derivative $k_{x_*} = \frac{d\omega^{x_*}}{d\sigma}$ is called the *Poisson kernel* for Ω. The most classical result here is Kellogg's theorem.

Kellogg's Theorem. *If $\partial\Omega \in C^{1+\alpha}, \alpha > 0$, then*

$$k_{x_*}(Q) = c_n \frac{\partial G(x_*, Q)}{\partial \vec{n}_Q}$$

and $\log k_{x_*} \in C^\alpha$.

Alt and Caffarelli proved a converse to Kellogg's theorem. A special case of their theorem is:

Alt-Caffarelli Theorem [AC]**.** *If $\partial\Omega \in C^1$ and if*

$$\frac{d\omega^{x_*}}{d\sigma} = k_{x_*}(Q) = c_n \frac{\partial G(x_*, Q)}{\partial \vec{n}_Q}$$

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satisfies $\log k \in C^\alpha$ *for some* $\alpha > 0$*, then* $\partial\Omega \in C^{1+\beta}$ *for some* $\beta = \beta(\alpha, n) > 0$.

Actually, Jerison [J] proved $\beta = \alpha$ in the Alt-Caffarelli theorem.

This book concerns the limiting forms of these two theorems as $\alpha \to 0$, either in the condition $\partial\Omega \in C^{1+\alpha}$ or in the condition $\log k \in C^\alpha$. If $H^n(\partial\Omega) < \infty$, the appropriate limit of C^α is the Sarason space $VMO(\sigma)$, which is defined by $f \in L^2(\sigma)$ and

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} \inf_{c\in R} \int_{B(Q,r)\cap\partial\Omega} |f(y) - c| \frac{d\sigma}{\sigma(B(Q,r))} = 0.$$

But if $H^n(\partial\Omega) = \infty$, the limiting condition must be expressed in terms of ω alone: it is that ω is an *asymptotically optimally doubling measure*, for all $x \in \Omega$,

$$\lim_{r\to 0} \frac{\omega^x(B(Q,\lambda r))}{\omega^x(B(Q,r))} = \lambda^n$$

uniformly in $Q \in \partial\Omega$.

2. The Venerable Case $n + 1 = 2$**.** This case is different because now Ω is a simply connected plane domain, so that there is a conformal map $g : \mathbb{D} \to \Omega$ from the unit disc $\mathbb{D}$ onto Ω and much is known about the relationships between $\partial\Omega$, ω and g. When $\partial\Omega$ is a Jordan curve, g extends continuously to $\partial\mathbb{D}$ and we have:

F. and M. Riesz Theorem. *If* $\partial\Omega$ *is a rectifiable Jordan curve, then the function* $g : \partial\mathbb{D} \to \partial\Omega$ *is absolutely continuous and gives the arc-length parameterization* σ *of* $\partial\Omega$,

$$\sigma(g(E)) = \int_E |g'| d\theta.$$

Moreover, $\log |g'| \in L^1(\partial\mathbb{D})$.

It follows that $\arg \varphi'$ is defined almost everywhere on $\partial\mathbb{D}$ and $\partial\Omega$ has unit normal $e^{i\theta} \frac{g'(e^{i\theta})}{|g'(e^{i\theta})|}$ at $d\sigma$ almost every $g(e^{i\theta}) \in \partial\Omega$. Consequently $\partial\Omega \in C^1$ if and only if $\arg g' \in C(\overline{\mathbb{D}})$ and $\partial\Omega \in C^{1+\alpha}$ if and only if $\arg g' \in C^{1+\alpha}(\overline{\mathbb{D}})$. If we choose $x_* = g(0)$, then ω^{x_*} is the transplant via g of $\frac{d\theta}{2\pi}$, while $d\sigma = |g'|d\theta$, so that

$$k = \frac{1}{|g' \circ g^{-1}|}.$$

A major goal of the work described in the book [B] is to liberate the theory of harmonic measure from its dependence on the conformal mapping g.

A closed curve $\Gamma \subset \mathbb{R}^2$ is a *chord-arc curve* or *Lavrentiev curve* if Γ is rectifiable and if there is a constant C such that for all $z, w \in \Gamma$ the shorter subarc $\Gamma(z, w) \subset \Gamma$ joining z and w satisfies

$$\sigma(\Gamma(z, w)) \leq C|z - w|$$

where σ is arc-length on Γ. One limiting result for $\alpha \to 0$ in the plane is:

Lavrentiev's Theorem [L]. *If* $\partial\Omega$ *is a chord-arc curve, then* $\arg g' \in BMO(\partial D)$ *and* $\log k \in BMO(\partial\Omega)$.

Here $f \in BMO(\sigma)$ if

$$||f||_{BMO} = \sup_{B(Q,r)} \inf_{c\in R} \int_{B(Q,r)} |f - c| \frac{d\sigma}{\sigma(B(Q,r))} < \infty$$

and σ is arc length on either ∂D or $\partial\Omega$.

The curve Γ is a *vanishing chord-arc curve* (or an *asymptotically smooth curve*) if

$$\lim_{|z-w|\to 0} \frac{\sigma(\Gamma(z,w))}{|z-w|} = 1.$$

A second $\alpha \to 0$ result for plane domains is:

Pommerenke's Theorem [P1]. *If $\partial\Omega$ is a rectifiable Jordan curve, then $\partial\Omega$ is a vanishing chord-arc curve if and only if $\arg g' \in VMO(\partial D)$ and if and only if $\log k \in VMO(\partial\Omega)$.*

Among the book's key theorems are higher dimensional versions of Lavrentiev's theorem and Pommerenke's theorem.

3. Nontangentially Accessible Domains. The NTA domains, introduced by Jerison and Kenig in the famous paper [JK], are a general family of domains for which many of the potential-theoretic properties of smooth domains continue to hold. The domain $\Omega \subset \mathbb{R}^2$ is an NTA domain if and only if $\partial\Omega$ is a quasicircle, and consequently $\partial\Omega$ need not have finite surface measure. By definition $\Omega \subset \mathbb{R}^{n+1}$ is an NTA domain if there are constants $R > 0$ and $M > 1$ such that the following three conditions hold:

(i) *Corkscrew Condition.* For all $Q \in \partial\Omega$ and all $0 < r < R$ there exists a ball $B(A, \frac{r}{M})$ with center $A = A(r, Q)$ such that

$$B(A, \frac{r}{M}) \subset \Omega \cap \{\frac{r}{M} \leq |X - Q| \leq r\}.$$

(ii) The complement $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ also satisfies the corkscrew condition (i).

(iii) *Harnack Chain Condition.* If $Q \in \partial\Omega$ and if $x_1, x_2 \in \Omega \cap B(Q, \frac{R}{4})$ and if $|x_1 - x_2| \geq 2^k \mathrm{dist}(x_i, \partial\Omega)$, then x_1 and x_2 are connected by a chain of Mk overlapping balls $B_j = B(p_j, r_j)$ such that $B(p_j, Mr_j) \subset \Omega$.

Suppose $\Omega \subset \mathbb{R}^{n+1}$ is an NTA domain. Then the Dirichlet problem is solvable and harmonic measure ω^x exists. But as noted above, $\partial\Omega$ may not have finite Hausdorff H^n measure, and we must use ω itself to describe potential-theoretic properties of Ω. Four important such properties of NTA domains are:

(a) *Boundary Regularity.* There exists $C > 0$ and $\beta > 0$ such that for all $Q \in \partial\Omega$ and all $0 < 2r < R$, if $u(x)$ is a positive harmonic function on $\Omega \cap B(Q, 2r)$ and u vanishes continuously on $B(Q, 2r) \cap \partial\Omega$, then for all $x \in \Omega \cap B(Q, r)$,

$$|u(x)| \leq C\Big(\frac{|x - Q|}{r}\Big)^{\beta} \sup_{\Omega \cap B(Q,2r)} u(y). \tag{2}$$

(b) *ω is a doubling measure.* If $r < \frac{R}{2}$, $Q \in \partial\Omega$ and $x \in \Omega \setminus B(Q, 2r)$, then for $0 < s < r$,

$$\omega^x(B(Q, 2s) \cap \partial\Omega) \leq C\omega^x(B(Q, s) \cap \partial\Omega)$$

where C depends only on R and M.

(c) *Comparison Principle.* Let $0 < Mr < R$, let $Q \in \partial\Omega$ and let $u(x)$ and $v(x)$ be positive harmonic functions on Ω which vanish continuously on $B(Q, rM) \cap \partial\Omega$. Then for all $x \in \Omega \cap B(Q, r)$,

$$C^{-1}\frac{u(A(r,Q))}{v(A(r,Q))} \leq \frac{u(x)}{v(x)} \leq C\frac{u(A(r,Q))}{v(A(r,Q))},$$

where $A(r, Q)$ is defined in the corkscrew condition (i) and C depends only on R and M.

(d) *Hölder Continuity of Kernels.* Fix $x_* \in \Omega$. Then for every $x \in \Omega$ and every $Q \in \partial\Omega$, the limit

$$K(x,Q) = \lim_{r\to 0} \frac{\omega^x(B(Q,r)\cap\partial\Omega)}{\omega^{x_*}(B(Q,r)\cap\partial\Omega)} \tag{3}$$

exists, and $Q \to K(x,Q)$ is Hölder continuous.

4. Reifenberg Flat Domains. A compact set $\Sigma \subset \mathbb{R}^{n+1}$ is (δ,R)-*Reifenberg flat* if for every $Q \in \Sigma$ and every $r \in (0,R)$, there exists an n-dimensional plane $L(r,Q) \subset \mathbb{R}^{n+1}$ such that $Q \in L(r,Q)$ and

$$\frac{1}{r}\mathrm{dist}_{\mathrm{H}}\big(\Sigma\cap B(Q,r), L(r,Q)\cap B(Q,r)\big) \le \delta,$$

where $\mathrm{dist}_{\mathrm{H}}$ denotes the Hausdorff distance between subsets of $\mathbb{R}^{n+1}$. The origin of this definition is the amazing theorem of Reifenberg [R] that if $\Sigma \subset \mathbb{R}^{n+1}$ is (δ,R)-Reifenberg flat and $\delta < \delta(n)$, then locally Σ is homeomorphic to an n-ball.

When $\Omega \subset \mathbb{R}^{n+1}$ is a domain, we say $\partial\Omega$ *separates* $\mathbb{R}^{n+1}$ if there exists $R > 0$ and $\delta > 0$ such that for all $Q \in \partial\Omega$ there exists an n-dimensional plane $\mathcal{L} = \mathcal{L}(R,Q)$ with normal vector $\vec{n}_{R,Q}$ such that $Q \in \mathcal{L}(R,Q)$ and

$$B(Q,R)\cap\{x = y + t\vec{n}_{R,Q} : y \in \mathcal{L},\ t > 2\delta R\} \subset \Omega,$$

and

$$B(Q,R)\cap\{x = y + t\vec{n}_{R,Q} : y \in \mathcal{L},\ t < -2\delta R\} \subset R^{n+1}\setminus\overline{\Omega}.$$

Finally, a domain $\Omega \subset \mathbb{R}^{n+1}$ is a (δ,R)-*Reifenberg flat domain* if $\partial\Omega$ separates $\mathbb{R}^{n+1}$ and if $\Sigma = \partial\Omega$ is (δ,R)-Reifenberg flat. In [KT1] Kenig and Toro proved that there is $\delta_0 = \delta_0(n)$ such that for $\delta < \delta_n$ every (δ,R)-Reifenberg flat domain in $\mathbb{R}^{n+1}$ is an NTA domain. This is Theorem 4.12 of [B]. Moreover, for Reifenberg flat domains smaller δ yields smaller $1-\beta$ in the boundary regularity estimate (2): Given $\epsilon > 0$ there exist $\delta = \delta(n,\epsilon)$ and $C = C(n,R,\epsilon) > 0$ such that if Ω is (δ,R)-Reifenberg flat, $Q \in \partial\Omega$, $0 < 2r < R$, and $u(x)$ is a positive harmonic function on $\Omega\cap B(Q,2r)$ such that $u(x)$ vanishes continuously on $B(Q,2r)\cap\partial\Omega$, then

$$|u(x)| \le C\Big(\frac{|x-Q|}{r}\Big)^{1-\epsilon} \sup_{\Omega\cap B(Q,2r)} u(y)$$

for all $x \in \Omega\cap B(Q,\frac{r}{2})$. This result from [KT3] is Lemma 5.1 in [B].

The domain $\Omega \subset \mathbb{R}^{n+1}$ is *Reifenberg flat with vanishing constant* if Ω is (δ,R)-Reifenberg flat for some $\delta < 1/(4\sqrt{2})$ and if the quotient (3) tends to 0 as $r \to 0$, uniformly in $Q \in \partial\Omega$. The (unbounded) region above the graph of the Zygmund function

$$\sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k\sqrt{k}}$$

is Reifenberg flat with vanishing constant, but its boundary is not locally rectifiable at any point. Vanishing constant Reifenberg flat domains have a beautiful description via doubling measures.

Theorem 1. [KT1] *If $\Omega \subset \mathbb{R}^{n+1}$ is Reifenberg flat with vanishing constant, then harmonic measure for Ω is asymptotically optimally doubling: for all $x \in \Omega$,*

$$\lim_{r\to 0} \frac{\omega^x(B(Q,\lambda r))}{\omega^x(B(Q,r))} = \lambda^n$$

uniformly in $Q \in \partial\Omega$.

The proof of Theorem 1, given in Chapter 6 of [B], is potential theoretic. It entails several comparison lemmas about harmonic measures.

The converse theorem, from [KT2], is a general result from geometric measure theory.

Theorem 2. [KT2] *Let μ be a doubling Radon measure supported on $\Sigma \subset \mathbb{R}^{n+1}$ and assume μ is also asymptotically doubling. If $n = 1, 2$, then Σ is Reifenberg flat with vanishing constant. If $n \geq 3$ and if Σ is $\delta(n)$-Reifenberg flat, then Σ is Reifenberg flat with vanishing constant.*

The proof of Theorem 2, given in Chapter 7 of [B], is a beautiful argument. It studies pseudo-tangent measures and reduces the problem to the Kowalski-Preiss theorem that the support of a uniform measure is either a plane or the product of a plane and a particular cone.

5. Chord Arc Domains. Chord arc domains are the rectifiable cousins of Reifenberg flat domains. A bounded open set $\Omega \subset \mathbb{R}^{n+1}$ has *finite perimeter* if the distribution $\nabla\chi_U$ is a vector-valued measure. The measure $\nabla\chi_U$ is supported on ∂U and the Gauss-Green theorem holds. See [EG]. By definition, $\Omega \subset \mathbb{R}^{n+1}$ is a (δ, R)-chord arc domain if Ω is a (δ, R)-Reifenberg flat domain, if Ω has finite perimeter, and if $\sigma = H^n|\partial\Omega$ satisfies

$$\sigma(B(Q,r)) \leq (1+\delta)\omega_n r^n$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Using a remarkable decomposition theorem for chord arc domains due to Semmes [S1], Kenig and Toro in [KT1] proved:

Theorem 3. *If Ω is a $(\delta, 2R)$-chord arc domain and $\delta < \delta(n)$, then*

(i) *the normal vector $\vec{n}$ to $\partial\Omega$ satisfies*

$$||\vec{n}||_{BMO(B(Q,R))} = \sup_{B(P,r)\subset B(Q,R)} \inf_{c\in R} \int_{B(P,r)} |\vec{n} - c| \frac{d\sigma}{\sigma(B(P,r))} \leq C\sqrt{\delta},$$

and

(ii) $\omega^x \ll \sigma \ll \omega^x$ *and the Poisson kernel satisfies* $\log k_x \in BMO$.

Theorem 3 is proved in Sections 4 and 6 of [B]. It gives the Lavrentiev theorem for $n+1 \geq 3$. The absolute continuity in (ii) was due originally to Semmes [S2] and David and Jerison [DJ].

The domain Ω is *chord arc with vanishing constant* if Ω is (δ, R)-chord arc for some δ and R, if Ω is Reifenberg flat with vanishing constant, and if

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega} \frac{\sigma(B(Q,r))}{\omega_n r^n} = 1.$$

The proof of Theorem 3 is quantitative, so that it yields

Corollary. *If $\Omega \subset \mathbb{R}^{n+1}$ is a chord arc domain with vanishing constant, then for all $x \in \Omega$, $\log k_x \in VMO(\partial\Omega)$.*

Conversely, in [KT3] Kenig and Toro proved:

Theorem 4. *Assume $\Omega \subset \mathbb{R}^{n+1}$ is a chord arc domain which is (δ, R)-Reifenberg flat for all R and for $\delta < \delta_0(n)$. If $\log k_x \in VMO(\partial\Omega)$, then Ω is chord arc with vanishing constant.*

Together the corollary and Theorem 4 give the Pommerenke theorem for $n+1 \geq 3$. The proof of Theorem 4, outlined in Section 7 of [B], uses more comparison theorems for harmonic measures on Reifenberg flat domains, a sharpened Semmes decomposition theorem and, most importantly, a careful "blow up" argument.

6. More Recent Results. The paper [DKT] gives an extension of Theorem 1 to Reifenberg flat sets of larger codimension. If $\Sigma \subset \mathbb{R}^m$ is a Reifenberg flat set of dimension $n < m$ with vanishing constant, then there exists an asymptotically optimal doubling measure having support Σ. The same paper shows that if μ is a positive measure supported on a δ-Reifenberg flat domain with $\delta < 1/(4\sqrt{2})$ and if μ is *n-uniform*: for all $Q \in \Sigma$ and $r < 1$,

$$\mu(B(Q, r) = \omega_n r^n,$$

then Σ is an analytic submanifold of $\mathbb{R}^{n+1}$. These and other related results are described in Section 8 of [B].

The paper [KT4] studies the situation when Ω is a two-sided NTA domain in $\mathbb{R}^{n+1}$, with harmonic measures ω_1 and ω_2 for points from the two sides of $\partial\Omega$. [KT4] proves that Ω is Reifenberg flat with vanishing constant if Ω is Reifenberg flat, if $\omega_1 \ll \omega_2 \ll \omega_1$ and if $\log(d\omega_2/d\omega_1) \in VMO(\omega_1)$, and proves that Ω is chord arc with vanishing constant if Ω is chord arc and $\log(d\omega_j/d\sigma) \in VMO(\sigma)$ for $j = 1, 2$. These results generalize the two-sides theorem from [BCGJ] for plane domains.

7. The Book [B]**.** Chapter 1 is a synopsis of the results treated later in the book, including their connections to previous work and including the relevant definitions. Chapter 2 gives a brief survey without proofs of the $n + 1 = 2$ case. Complete proofs and much more information about this different case can be found in books by the reviewer and D. Marshall [GM] or by Pommerenke [P2]. (Incidentally, the Notes to Chapter 2 have a reference error; paper [BJ] does include a complete characterization of domains for which $\log g' \in BMO(\partial D)$.) Chapter 3 is a survey without proofs of the basic results about NTA domains, A_∞ weights, and the spaces BMO and VMO. Chapters 4 and 5 discuss the geometry and potential theory of Reifenberg flat domains and of chord arc domains. Here many proofs are included. Chapter 6 treats the Geometry $\Longrightarrow$ Potential Theory results, including Theorem 1 and Theorem 3 above. Complete proofs are presented and each hard proof is preceded by a helpful "sketch of proof". Chapter 7 turns to the Potential Theory $\Longrightarrow$ Geometry theorems, including Theorem 2 and Theorem 4 above. The proofs of both are sketched, Theorem 2 is completely proved, but the reader is referred to [KT3] for the details of the lengthy proof of Theorem 4. Chapter 8 surveys further results mostly about the geometry of Reifenberg flat subsets of higher codimension.

This book is a good introduction to an exciting new research area on the interface of harmonic analysis and geometric measure theory. The book is very well written, with clear explanations and useful pictures. It negotiates a fine compromise between brevity and detail as it presents a subject that is necessarily somewhat technical.

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