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Extremum problems for eigenvalues of elliptic operators, by Antoine Henrot, Birkhäuser Verlag, Basel–Boston–Berlin, x + 202 pp., US$49.95, ISBN 978-3-7643-7705-2

"If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle," wrote Lord Rayleigh in his book *The Theory of Sound* ([R], vol. 1, §210). A membrane is a planar domain Ω, and its principal tone is $\sqrt{\lambda_1}$, with λ_1 being the smallest number for which the problem

$$\Delta u + \lambda u = 0, \quad u(x) = 0 \text{ when } x \in \partial\Omega, \tag{1}$$

has a non-trivial solution $u = u(x, y)$. Here, of course, Δ is the Laplace operator, $\partial\Omega$ is the boundary of Ω. Eigenvalues of the problem (1), enumerated in the increasing order, are usually denoted by $\lambda_k(\Omega)$. For mathematicians, it took close to 50 years to actually prove that, out of all domains $\Omega \subset \mathbb{R}^n$ of given volume, $\lambda_1(\Omega)$ is the smallest for the ball (the Farber–Krahn theorem). Another sentence from the same book, "when the edges [of a plate] are clamped, the form of the gravest pitch is doubtless the circle," became a theorem after more than 100 years. Nadirashvili proved that out of all planar domains of given area, the smallest eigenvalue of the bi-Laplacian Δ^2 with the Dirichlet boundary conditions is minimal for the circle. Ashbaugh and Benguria extended the theorem to domains in $\mathbb{R}^3$; for higher dimensions it is still an open problem.

Another problem is to study possible values for the gravest tone of a free membrane. This is the square root of $\mu_1(\Omega)$, the smallest positive eigenvalue of the Neumann Laplacian in Ω. The equation is the same as in (1), but the boundary condition is different: the normal derivative of u vanishes on $\partial\Omega$. For a rectangle, $\mu_1(\Omega) = \pi^2/l^2$ where l is its longest side, so in the class of domains of given area, μ_1 can be made arbitrarily close to 0: take a long, thin rectangle. The relevant question is how big $\mu_1(\Omega)$ can be. Szegö proved that out of all simply connected planar domains of given area, μ_1 is the largest for a circle. Later, Weinberger extended the theorem to arbitrary domains in $\mathbb{R}^n$.

The number of eigenvalues is infinite, so there is an infinite number of questions to be asked. What is the smallest possible value of $\lambda_k(\Omega)$? What is the largest possible value of $\mu_k(\Omega)$? Then one can study different combinations of eigenvalues. One can impose different constraints: fix, say, the diameter, not the area. One can impose different boundary conditions. One can take different operators. One can consider smaller classes of domains, or one can study domains on manifolds. Not all the questions are equally interesting, but almost all of them are difficult. A naive conjecture would be that, out of all domains of given volume, λ_k is the smallest for the ball. This is already wrong when $k = 2$: the answer is the disjoint union of two identical balls. If one insists on a domain to be connected, then there is no minimizer. Bucur and Henrot proved that a minimizer exists for λ_3. It is not known what the shape of the minimizer is; a conjecture is the ball in dimensions 2 and 3, the disjoint union of two identical balls in higher dimensions. It is still not known whether a minimizer for λ_k, $k \geq 4$, exists. Going in a different direction,

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one can ask what N-gon of given area has the smallest value for λ_1. The answer is almost obvious – of course, the regular N-gon. However, for $N \geq 5$, it is still an open problem.

The variational formulation of the eigenvalue problem in terms of the Rayleigh quotient $R[u] = \int |\nabla u|^2 dx / \int |u|^2 dx$ lies at the center of the whole business. In particular, $\lambda_1(\Omega)$ is the minimal value of the Rayleigh quotient over all functions that are equal to 0 on the boundary of Ω. Let me quote Rayleigh one more time: "... much stress was laid upon the establishment of general theorems by means of Lagrange's method, and I am more than ever impressed with the advantages of this procedure." To prove the Farber–Krahn theorem, for example, what one needs to do is construct for every function $u(x)$ in Ω that vanishes on $\partial\Omega$ another function $u^*(x)$ in the ball B of the same volume, vanishing on ∂B, for which $R[u^*] \leq R[u]$. This can be done by using spherical rearrangement: $u^*(x)$ is a decreasing function of the distance to the center of B such that the sets $\{x \in \Omega : |u(x)| > t\}$ and $\{x \in B : u^*(x) > t\}$ have the same measure for all values of t. The denominators in the Rayleigh quotients for u and u^* are the same, and for the numerators there is Pólya's inequality

$$\int_B |\nabla u^*(x)|^2 dx \leq \int_\Omega |\nabla u(x)|^2 dx.$$

One does not have to worry whether or not the function $u(x)$ is positive: replacing $u(x)$ by $|u(x)|$ does not result in any change in the Rayleigh quotient. For the clamped plate problem, the situation is radically different. The numerator in the Rayleigh quotient for that problem is $\int |\Delta u|^2 dx$, and for a typical function $u(x)$ that changes sign, $\Delta|u|$ is a distribution, not a square integrable function. For the proof of the Farber–Krahn theorem one can actually make the spherical rearrangement of the first eigenfunction only, and the first eigenfunction of the fixed membrane problem does not change its sign. However, the first eigenfunction of the clamped plate problem may well be of variable sign; this is what makes the problem more difficult.

The simplest expressions that involve more than one eigenvalue are probably the ratio of the second and the first eigenvalues and the difference between them. In the case of the fixed membrane problem (1), these are λ_2/λ_1 and $\lambda_2 - \lambda_1$. The advantage of the ratio is that it is scale invariant, and one does not have to impose any additional constraints for a minimization/maximization problem. This is part of a broader question: what is the range of the mapping $\Omega \mapsto \{\lambda_k(\Omega)\}$? Finding the complete answer is probably hopeless, but we know that $\lambda_1 < \lambda_2$ (the ground state of a connected domain cannot be multiple), we know that $\lambda_k \sim Ck^{2/n}$ (Weyl's law), and the range of values of λ_2/λ_1 is one additional constraint on the sequence $\{\lambda_k\}$. By taking long, thin rectangles, one can make λ_2/λ_1 arbitrarily close to 1. Ashbaugh and Benguria proved that $\lambda_2(\Omega)/\lambda_1(\Omega)$ is the largest when Ω is, not surprisingly, a ball. For the difference $\lambda_2(\Omega) - \lambda_1(\Omega)$ (it is called the fundamental gap), the problem is to find its best possible lower bound for convex domains of given diameter d. The conjecture is that the bound is $3\pi^2/d^2$, and it is attained on a sequence of thin rectangles (rectangular parallelepipeds) with the longest side approaching d.

One can study expressions that incorporate several eigenvalues or even all of them. One of the most popular functions is the heat trace

$$h_\Omega(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)}.$$

Luttinger proved that if B is the ball of the same volume as Ω, then $h_\Omega(t) \le h_B(t)$ for all positive values of t. In the proof he used spherical rearrangement and the Feynmann–Kac formula. Actually, he made the comparison between the heat trace for a Schrödinger operator with a non-negative potential $V(x)$ that grows at infinity sufficiently fast and the heat trace for the Schrödinger operator $-\Delta + V^*(x)$, where $V^*(x)$ is the increasing spherical rearrangement of $V(x)$. Then one takes $V(x) = 0$ in Ω and $V(x) = \infty$ outside Ω. The second popular function is the ζ-function,

$$\zeta_\Omega(z) = \sum_{j=1}^{\infty} \lambda_j(\Omega)^{-z}. \tag{2}$$

The series (2) converges absolutely when $\Re z > d/2$, so $\zeta_\Omega(z)$ is holomorphic in that half-plane. The heat trace and the zeta function are related:

$$\zeta_\Omega(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} h_\Omega(t)dt.$$

In particular, this implies that $\zeta_\Omega(s) \le \zeta_B(s)$ when s is a real number and $s > n/2$. In other words, the sums of powers of reciprocals of $\lambda_k(\Omega)$ are smaller than similar sums for the ball of the same volume.

Conformal methods play an important role in the study of two-dimensional problems. Szegö used them to prove that the disc minimizes $\mu_1(\Omega)^{-1} + \mu_2(\Omega)^{-1}$ in the class of simply connected planar domains of given area (recall that μ_j's are positive eigenvalues of the Neumann Laplacian). Hersch proved that the smallest positive eigenvalue of the Laplace-Beltrami operator on a two-sphere cannot exceed that of the operator for the round metric of the same area. The crucial observation is that the numerator in the Rayleigh quotient, $\int |\nabla u|^2 dx$, is conformally invariant when the dimension equals 2. Another, somewhat more sophisticated, problem where the conformal methods are used is the study of the determinant. It is a classical theorem due to Seeley that the zeta-function (2) extends to a meromorphic function in the whole complex plane; moreover, $z = 0$ is its regular point. The determinant of the Laplacian on a Riemannian manifold is defined as $\exp(-\zeta'(0))$. Osgood, Phillips, and Sarnak proved in [OPS] that in a given conformal class of metrics on a surface, the metric of constant curvature has maximal determinant. In particular, this gives a proof of the uniformization theorem.

There are several books on isoperimetric inequalities that touch on spectral problems. First the famous book [PS] by Pólya and Szegö, then a book [B] by C. Bandle. Both books treat variational problems for different physical quantities associated to a domain: natural frequencies is an important example, but it is not the only example. The book under review is probably the first book with an exclusive emphasis on variational problems for eigenvalues of elliptic operator. The literature on the subject is extensive, and there has been significant progress made in the last 25 years, so the idea of writing a book was more than justified.

The book covers many variational problems, though it is not encyclopedic. The author formulates 30 open problems; some of them are well known, some of them are more obscure. It is a pity that conformal methods are mentioned only in passing, but this is one of the choices that the author had to make. Unfortunately, the author chose to omit proofs of many important facts. Even Pólya's inequality that lies at the heart of the whole business is given without proof. In many cases, the proofs are just sketched. Sometimes they are a bit murky. One can find incorrect statements in the book. It is easy to construct a counterexample to the statement of Remark 2.3.5. In the "proof", the author mixes pointwise convergence with uniform convergence. Theorem 6.3.1 claims that the disk minimizes the sum of reciprocals of $\lambda_j(\Omega)$ over all planar domains Ω of given area. This sum is always infinite. The theorem is attributed to Luttinger. There is nothing like that in [L]. Some of the terminology, like "a basis of a triangle", "a mediator of a side of a triangle", or "derivable function", is unorthodox. Strong convergence of operators is called "simple convergence", and convergence in norm is called "strong convergence".

I think that the book will be useful for the experts. It is convenient to have many facts and open problems collected in one place. I would be very cautious in recommending the book to somebody who wants to learn the material. For that purpose, I would still recommend [PS], [B], and the original papers.

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