BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, April 2008, Pages 303–307
S 0273-0979(08)01214-7
Article electronically published on February 14, 2008

SELECTED MATHEMATICAL REVIEWS

related to the paper in this section by
GRAY AND MICALLEF

MR0173993 (30 #4200) 53.04

Nitsche, Johannes C. C.

On new results in the theory of minimal surfaces.

Bull. Amer. Math. Soc. **71** (1965), 195–270.

In this expository article a rather impressive array of evidence is gathered showing that minimal surfaces still furnish a rich source of results and problems, despite the fact that so many generations of mathematicians have considered them and so much progress was made in the 1930's by Douglas, Rado, and others. The account is sufficiently comprehensive to be of real value to experts in the subject, and at the same time is sufficiently non-technical and provides enough historical background to interest a much wider audience.

Among the "new results" surveyed, a considerable number concern nonparametric minimal surfaces, i.e., surfaces of the type $z = z(x, y)$ satisfying the minimal surface equation. Among the topics included are the following: nonsolvability of the minimal surface equation over a nonconvex domain with arbitrary Dirichlet boundary data, removability of singularities, Bernstein's theorem for complete minimal surfaces, and a priori estimates for first order partial derivatives and for the Gauss curvature.

Among the topics covered about minimal surfaces given parametrically are the question of branch points, analyticity of minimal surfaces across analytic boundary arcs, doubly connected minimal surfaces and complete minimal surfaces. A list of 50 research problems is included.

Up to now comparatively few results about minimal k-dimensional surfaces have been proved when $k > 2$. Nevertheless, it would have been desirable in such a comprehensive survey to have a better account of the recent important work in this direction of Reifenberg [Acta Math. **104** (1960), 1–92; MR0114145 (22 #4972); Ann. of Math. (2) **80** (1964), 1–14; MR0171197 (30 #1428); ibid. (2) **80** (1964), 15–21; MR0171198 (30 #1429)] and of De Giorgi ["Frontiere orientate di misura minima", Sem. Mat. Scuola Norm. Sup. Pisa, 1961]. It must also be pointed out that in its strongest form the Plateau problem is to find a minimal surface S spanning a given boundary curve C such that S is without singular points (i.e., branch points). In this form, the problem was never solved until Reifenberg's Acta Math. paper [loc. cit.].

From MathSciNet, January 2008

W. H. Fleming

MR0428181 (55 #1208a) 49F22 (58A99)

Taylor, Jean E.

The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces.

Ann. of Math. (2) **103** (1976), *no.* 3, 489–539.

MR0428182 (55 #1208b) 49F22 (58A99)

Taylor, Jean E.

The structure of singularities in solutions to ellipsoidal variational problems with constraints in R^3.

Ann. of Math. (2) **103** (1976), *no.* 3, 541–546.

More than 100 years ago Plateau, in his experimental study of compound soap films and soap bubbles, recorded the following "laws" governing their geometric structure: A compound soap-film-like surface always consists of smooth interfaces which are joined together along "singular arcs" where three interfaces meet at equal angles and at "singular vertices" where six interfaces and four singular arcs are brought together at equal angles. Using a mathematical model for surfaces which is sufficiently general to cover (the mathematical idealizations of) many actual compound surfaces governed by surface tension and additional forces the author proves for the first time that the branching types described by Plateau are indeed the only ones which can occur (and, in fact, do occur in the mathematical model). Moreover, the local structure of the model surfaces near their singular set is completely analyzed with the result that, near a singular point, every surface can be obtained from its tangent cone at that point by a Hölder continuously differentiable diffeomorphism of the ambient space. The results of these papers are certainly among the most significant contributions to the mathematical theory of soap films in this century. (They had been announced previously [Bull. Amer. Math. Soc. **81** (1975), no. 6, 1093–1095; MR0388223 (52 #9060)].)

The mathematical model for compound surfaces used are the (ξ,δ) minimal sets of F. J. Almgren, Jr. [Mem. Amer. Math. Soc. **4** (1976), no. **165** (1976); MR0420406 (54 #8420)]. Here δ is a positive constant, $\xi\colon \mathbf{R}_+ \to \mathbf{R}_+ \cup \{0\}$ is a function of type $\xi(r) = Cr^\alpha$ with $C \geq 0$, $\alpha > 0$, and a set $S \subset \mathbf{R}^3$ is (ξ,δ) minimal with respect to a closed "boundary" (or "obstacle") $B \subset \mathbf{R}^3$ if S satisfies some technical conditions (e.g., S is $(\mathcal{H}^2, 2)$ rectifiable and equals the closure relative to B of its $\mathcal{H}^2$-essential points) and has the following property: Whenever $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is Lipschitz and equals the identity outside some ball W in $\mathbf{R}^3 \sim B$ of radius $r < \delta$, then $\mathcal{H}^2(S \cap W) \leq (1 + \xi(r))\mathcal{H}^2(\varphi(S \cap W))$. Here, $\mathcal{H}^2$ denotes two-dimensional Hausdorff measure on $\mathbf{R}^3$. Thus, inside balls of sufficiently small radius r the surface S is "nearly area-minimizing" in the sense that passing to the image of S by a Lipschitz map φ as above cannot decrease area by a factor larger than $1 + \xi(r)$. The property of being (ξ,δ) minimizing is considerably weaker than any local area-minimizing property of S. On the other hand, it is not true, in general, that the image of a mapping from some planar domain into $\mathbf{R}^3$ which locally minimizes the area integral is (ξ,δ) minimal for some ξ and δ; hence the results of the papers under review do not apply to such "classical minimal surfaces". (Indeed, the image of an area-integral-minimizing mapping which is not an embedding always has transversal self-intersections, a kind of singularity that cannot occur in actual compound soap-films.) Almgren [op. cit.] has shown that the solutions to a variety of geometric and physical variational problems for surfaces of arbitrary dimension and codimension are (ξ,δ) minimal with suitable ξ and δ and he has proved regularity almost everywhere of (ξ,δ) minimal sets. His theory includes the description, by variational conditions, of compound soap films subject to exterior forces like gravity and to additional conditions such as maintaining the volume enclosed by certain interfaces.

The results of the first paper completely clarify the local structure of two-dimensional (ξ, δ) minimal sets in $\mathbf{R}^3$. To state them precisely, let D denote the closed unit disc in $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$, denote by Y the union of the half-disc $D \cap \{(x_1, x_2, x_3)\colon x_2 \geq 0\}$ with its rotations about the x_1-axis by 120° and by 240° and define T as the intersection of the closed unit ball in $\mathbf{R}^3$ with the infinite cone from the origin of $\mathbf{R}^3$ through the one-skeleton of a regular tetrahedron centered at the origin. Main result: If S is (ξ, δ) minimal with respect to B and $p \in S$, then there exists a Hölder continuously differentiable diffeomorphism f from the closed unit ball in $\mathbf{R}^3$ onto a closed neighborhood N of p in $\mathbf{R}^3 \sim B$ such that either $S \cap N = f(D)$ or $S \cap N = f(Y)$ or $S \cap N = f(T)$; in the latter cases f can be chosen to be conformal on the singular set of $Y[T]$.

The first step in the proof is the classification of possible tangent cones to S. It is shown that rectifiable tangent cones exist at every point $p \in S$ (not necessarily unique, a priori). From (ξ, δ) minimality of S one infers that tangent cones are area-minimizing with respect to their intersection with the unit sphere in $\mathbf{R}^3$. (All tangent cones are translated to the origin and intersected with the closed unit ball.) This fact, in turn, implies that the intersection of a tangent cone with the unit sphere is a "120°-net", i.e., a union of great circle segments intersecting three at a time at a finite number of points and with angles 120° to each other at the intersection points. The 120°-nets on the unit sphere have been classified; up to isometry they are (1) a single great circle, (2) three half great circles having common end points, (3) the one-skeleton of a regular spherical tetrahedron, (4) [(5)] the one-skeleton of a spherical cube [regular dodecahedron] (6) [(7)] the one-skeleton of a spherical prism over a 120° triangle [regular pentagon], (8), (9), (10) three "irregular" ones having quadrangles and non-equilateral pentagons. {The history of this classification is quite interesting. The first attempt was made by Lamarle in connection with investigations of the shape of soap films as early as 1864; however, the tenth net eluded Lamarle. The correct classification was given by Heppes in 1964 who was apparently unaware of its relation to the geometry of soap films.} It turns out that only the cones corresponding to the first three nets can be area-minimizing, i.e., only rotations of D, Y and T can occur as tangent cones. At the end of the first paper it is shown that T must indeed occur as a tangent cone to some (ξ, δ) minimal set and is hence area-minimizing. The corresponding fact for Y has been proved earlier by the author and is obvious for D.

The classification of tangent cones clarifies the infinitesimal structure of S near a singular point p. The main result of the first paper is, however, the determination of the local structure of S near p. This amounts to a proof of regularity of S near the singular set and requires deep methods which can only be touched upon here. The idea is to show that Almgren's estimates [op. cit.] for the Hölder norm of the derivatives of suitable non-parametric representations of S near a regular point q (i.e., the tangent cone at q is a disc) hold uniformly as q approaches the singular point p. The basic ingredient one needs for this purpose is an estimate of type $\mathbf{e}_p(r) \leq cr^\mu$ with positive c and μ for the "excess" $\mathbf{e}_p(r) = r^{-2}\mathcal{H}^2(S \cap \mathbf{B}(p,r)) - d_p$, where $\mathbf{B}(p,r)$ is the ball of radius r centered at p and $d_p = \lim_{r\to 0+}[\mathcal{H}^2(D \cap B(0,r))]^{-1}\mathcal{H}^2(S \cap \mathbf{B}(p,r))$. The uniqueness of tangent cones follows immediately from such an inequality. The estimate for the excess is derived from the "epiperimetric inequality" which is similar to previous inequalities by E. Reifenberg [Ann. of Math. (2) **80** (1964), 1–14; MR0171197 (30 #1428)] and the author [Invent. Math. **22** (1973), 119–139;

MR0333903 (48 #12225)]. The epiperimetric inequality is proved by an intricate contradiction argument. Therefore, one can only conclude the existence of some positive μ and c such that $\mathbf{e}_p(r) \leq cr^\mu$ and, as a consequence, one cannot give an explicit lower bound for the Hölder exponent of the derivative of the diffeomorphism f in the main result.

In the second paper it is shown that the results of the first paper extend to ellipsoidal integrands on $\mathbf{R}^3$, i.e., to the area integrand corresponding to an arbitrary Hölder continuous Riemannian metric on $\mathbf{R}^3$.

{The fact that the cones corresponding to the 120°-nets (4)-(10) are not area-minimizing can also be verified experimentally by dipping wire models of the nets into a soap solution. Only for nets (1), (2) and (3) will the resulting compound soap film be a cone. The configurations one obtains in the other cases are indicated by figures in the first paper. Beautiful illustrations and photographs of soap films, in particular of the ten 120°-nets and the soap films corresponding to them, can be found in the article by F. J. Almgren, Jr. and the author [Scientific American **235** (1976), no. 1, 82–93]. The most striking soap film configurations the reviewer has ever seen arise when one blows a soap bubble into one of the 120°-wire-nets, such that the net comes to lie on the surface of the bubble, and then extracts the air slowly; the configuration will, to a certain degree, approach the cone corresponding to the net and, in case this cone is not area-minimizing, it will switch to a quite different shape as soon as the remaining enclosed volume drops below a certain critical value.}

From MathSciNet, January 2008

Klaus Steffen

MR0775682 (87a:58041) 58E12 (49F10 53A10)

Giusti, Enrico

Minimal surfaces and functions of bounded variation.

Monographs in Mathematics, 80.

Birkhäuser Verlag, *Basel*, 1984. xii+240 *pp*. *ISBN* 0-8176-3153-4

This book deals with Plateau's problem, which is to find a hypersurface of least area spanning a given boundary. It was only in 1930/31 that a solution of this old problem was found by Douglas and Radó for surfaces in $\mathbf{R}^3$, and it took another 30 years until the higher-dimensional case was attacked by means of measure-theoretic methods. Pioneers of these new methods were De Giorgi, Reifenberg, Fleming, Almgren and Federer.

The first part of this book is devoted to the representation of De Giorgi's approach to the parameteric minimal surfaces, while the second part deals with non-parametric surfaces. De Giorgi defined hypersurfaces admissible for the Plateau problem as boundaries of Caccioppoli sets. These are Borel sets E in $\mathbf{R}^n$ with the property that the distributional derivative $D\varphi_E$ of the characteristic function φ_E is a Radon measure of locally bounded total variation. The perimeter of a Caccioppoli set E in a bounded open set Ω in $\mathbf{R}^n$ is defined as the total variation of $D\varphi_E$ on Ω:

$$P(E,\Omega) = \int_\Omega |D\varphi_E| = \sup\{\int_E \operatorname{div} g\colon\ g \in C^1_c(\Omega,\mathbf{R}^n), \quad |g| \leq 1\}.$$

With this weakened notion of "surface" the Plateau problem is easy to solve in the following sense: Theorem (existence of a weak solution): Let a bounded open set

Ω and a Caccioppoli set L in $\mathbf{R}^n$ be given. Then in the class of Caccioppoli sets $\{E\colon E\backslash\Omega = L\backslash\Omega\}$ there exists a set of least perimeter in Ω, called a minimal set in Ω.

The proof of the regularity almost everywhere, however, requires rather hard work. For this the notion of the reduced boundary $\partial^* E$ of a set E is introduced as the set of points $x \in \mathbf{R}^n$ for which there exists a generalized unit normal vector $\nu(x)$ as the limit as $r \to 0$ of the vectors $\nu_r(x) = \int_{B(x,r)} D\varphi_E / \int_{B(x,r)} |D\varphi_E|$. Theorem (partial regularity): For any minimal set E in Ω the reduced boundary $\Omega \cap \partial^* E$ is an analytic hypersurface and the singular set $\Omega \cap (\partial E \backslash \partial^* E)$ has $(n-1)$-dimensional Hausdorff measure 0.

The tools for the proof of these results are provided in the first 8 chapters. In Chapters 1–4 properties of functions of bounded variation are derived, in particular compactness, semicontinuity of the total variation, approximation by smooth functions and traces. The regularity proof covers Chapters 5–8. Chapters 9–11 are devoted to the investigation of the singular set. For every point $x_0 \in \partial E$ there exists the tangent cone C as L_1^{loc}-limit of a suitable sequence of the expanding sets $E_t = \{x \in \mathbf{R}^n\colon x_0 + t(x - x_0) \in E\}$ as $t \to 0$. The tangent cone is minimal if E is minimal in a neighbourhood of x_0, and x_0 belongs to the reduced boundary if and only if ∂C is a hyperplane.

Theorem (Almgren, Simons): Let F be a cone in $\mathbf{R}^n$ whose boundary is smooth outside the vertex. If the first and second variations of area of ∂F satisfy $\delta A = 0$ and $\delta^2 A \geq 0$, then ∂F is a hyperplane for $n \leq 7$. $n = 7$ is optimal (Bombieri, De Giorgi, Giusti). Therefore, for $n \leq 7$ the boundary $\Omega \cap \partial E$ of every minimal set E in $\Omega \subset \mathbf{R}^n$ is analytic. This result is completed by another theorem (Federer): The s-dimensional Hausdorff measure of the singular set vanishes for any $s > n-8$.

Part 2 of the book deals with nonparametric minimal surfaces, i.e. graphs of functions $u\colon \Omega \subset \mathbf{R}^n \to \mathbf{R}$ in $\mathbf{R}^{n+1}$. By minimizing the area integrand on the space of functions with bounded Lipschitz constant, by using barriers and a gradient estimate, the following result is obtained (Chapters 12, 13): Theorem (M. Miranda): If $\partial\Omega$ has nonnegative mean curvature, the nonparametric Plateau problem has a unique solution, which is as smooth as the data allow.

In Chapters 14–16 the nonparametric problem is also attacked by using functions of bounded variation, whereby more general boundaries can be allowed. Finally Chapter 17 is devoted to the Bernstein problem. Bernstein's famous theorem from 1915 for entire minimal surfaces in $\mathbf{R}^3$ has the following extension. Theorem (Fleming): For $n \leq 7$ the boundary of every minimal set in $\mathbf{R}^n$ is a hyperplane. Again, $n = 7$ is optimal.

These remarkable results arise from a most fruitful period in the theory of higher-dimensional minimal surfaces. This book gives at first a comprehensive and self-contained representation of this important development. The reading requires only a "fairly good knowledge of general measure theory and some familiarity with the theory of elliptic partial differential equations". It is facilitated by a well-balanced notation. In its entirety, the book is, in the opinion of the reviewer, a masterpiece of clear (and aesthetic) representation of highly nontrivial analysis.

From MathSciNet, January 2008

Helmut Kaul