BOOK REVIEWS

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Valuations, orderings and Milnor K-theory, by Ido Efrat, Mathematical Surveys and Monographs, vol. 124, American Mathematical Society, Providence, RI, 2006, xiv+288 pp., US$60.00, ISBN 978-0-8218-4041-2

The book is divided into four parts: Abelian Groups, Valuations and Orderings, Galois Theory, and K-Rings. In my review I will concentrate on the last part, where, it seems to me, the essential motivation for writing the book is to be found. I will of course, at the same time, present my own views as to why this particular mix of topics is interesting.

In the book one encounters most of the things one might expect to find in an introductory book on valuation theory, but this is combined and integrated with the study of orderings (insofar as this is reasonable and convenient), and there is an emphasis on facets of the subject that have been developed or have taken on new significance recently, because of their connection to Milnor K-theory (and, by extension, to quadratic form theory and Galois cohomology). The presentation is self-contained, making the book easily accessible to students.

The Milnor ring of a field F [Mi] is the graded ring $K_*(F)$ defined as follows: $K_*(F) = \oplus_{r=0}^{\infty} K_r(F)$ where $K_0(F) = \mathbb{Z}$, $K_1(F) =$ the group $F^\times$, but written additively (i.e., $K_1(F) = \{\ell(a) \mid a \in F^\times\}$ with $\ell(a) + \ell(b) := \ell(ab)$), and $K_r(F) =$ the tensor product over $\mathbb{Z}$ of r copies of $K_1(F)$ modulo the subgroup generated by all $\ell(a_1) \otimes \cdots \otimes \ell(a_r)$ with $1 = a_i + a_j$ for some $i \neq j$, for $r \geq 2$. In the book, the distinguished element $\epsilon := \ell(-1) \in K_1(F)$ is considered to be an integral part of the structure. Multiplication on $K_*(F)$ is induced by $\otimes$. This satisfies

$$\begin{cases} \ell(a)^2 = \epsilon\ell(a) = \ell(a)\epsilon \quad \text{and} \\ \ell(a)\ell(b) = -\ell(b)\ell(a), \end{cases}$$

for all $a, b \in F^\times$. Because of the way $K_*(F)$ is described, in terms of generators and relations, $K_*(F)$ is completely determined by the groups $K_i(F)$, $i = 1, 2$, the multiplication $K_1(F) \times K_1(F) \to K_2(F)$, and the distinguished element $\epsilon \in K_1(F)$.

In the book, certain 'relative' Milnor rings $K_*(F)/S$ are considered, where S is a subgroup of $F^\times$. These relative Milnor rings can be understood in the general context of multifields introduced just recently in [M2]. A multifield is a structure just like a field, except that the addition is multivalued. If F is a field (or multifield) and S is a subgroup of $F^\times$, we can form the quotient multifield $F/_m S$. Declare $a, b \in F$ to be equivalent if $as = bt$ for some $s, t \in S$. Denote the equivalence

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class of $a \in F$ by $\overline{a}$. $F/_mS$ is just the set of equivalence classes $\{\overline{a} \mid a \in F\}$ with multifield structure induced by the field structure on F, i.e., $0 = \overline{0}$, $1 = \overline{1}$, $-\overline{a} = \overline{-a}$, $\overline{a} + \overline{b} = \{\overline{c} \mid cs = at + bu \text{ for some } s, t, u \in S\}$, $\overline{a}\overline{b} = \overline{ab}$. (If F is only assumed to be a multifield, one should replace the condition $cs = at + bu$ by the corresponding multifield condition $cs \in at + bu$.) As a set, $F/_mS$ is just $F^\times/S \cup \{0\}$. If F is any multifield, one can define the Milnor ring $K_*(F)$ exactly as before. (Just replace the condition $1 = a_i + a_j$ in the definition by the corresponding multifield condition $1 \in a_i + a_j$.) Multifields form a category in a natural way and $F \rightsquigarrow K_*(F)$ defines a functor on this category. (A suitable target category, the so-called category of κ-structures, is defined in the book.) For any subgroup S of $F^\times$, the relative Milnor ring $K_*(F)/S$ is just the Milnor ring $K_*(F/_mS)$.

There is special interest in the subgroups $(F^\times)^n := \{a^n \mid a \in F^\times\}$, $n \geq 1$. As one might expect, the homogeneous part of degree r of $K_*(F)/(F^\times)^n$ is just $K_r(F)/nK_r(F)$ if $r \geq 1$. This is *not* true however for $r = 0$ (since, by definition, the homogeneous part of degree zero is always $\mathbb{Z}$).

One sees the connection between orderings, valuations and quadratic forms already in the classical work of Hasse and Minkowski on quadratic forms over number fields (finite extensions of $\mathbb{Q}$). The paper of Pfister [Pf] exploits Artin-Schreier theory to understand the relationship between orderings and quadratic forms over an arbitrary field F, $\operatorname{char}(F) \neq 2$. One typically studies quadratic forms over a field F, $\operatorname{char}(F) \neq 2$, by studying the Witt ring $W(F)$ of all Witt equivalence classes of (non-degenerate) quadratic forms over F; see [Pf] or [W]. At the same time, it is well-known that the ring $W(F)$ carries precisely the same information about F as the relative Milnor ring $K_*(F)/(F^\times)^2$: If F_1, F_2 are fields, $\operatorname{char}(F_i) \neq 2$, $i = 1, 2$, then $W(F_1) \cong W(F_2)$ as rings iff $K_*(F_1)/(F_1^\times)^2 \cong K_*(F_2)/(F_2^\times)^2$ as graded rings with distinguished element.[1] It is important to say, though, that it is often a highly non-trivial problem to translate results about $W(F)$ into results about $K_*(F)/(F^\times)^2$ (or the other way around).

There is also the connection with Galois cohomology: Fix a positive integer n and a field F with $\operatorname{char}(F) \nmid n$. Denote by F_{sep} the separable closure of F, μ_n the group of n-th roots of unity in F_{sep} and G_F the Galois group of F_{sep} over F. Recently, Voevodsky [V2] announced a proof of the Block-Kato conjecture: The natural group homomorphism

$$K_r(F)/nK_r(F) \to H^r(G_F, \mu_n^{\otimes r})$$

induced by the cup product is an isomorphism, for all $r \geq 1$. When $n = 2$ this is the Milnor conjecture, which was proved in [V1]. The case $r = 1$ is immediate from Hilbert's 'Satz 90'. The case $r = 2$ was proved earlier in [M-S]. Suppose now that $n = p$, p prime, and $\mu_p \subseteq F^\times$ (this includes the case $n = 2$). Fixing a generator ξ_p of μ_p, one sees that μ_p is identified with $\mathbb{Z}/p\mathbb{Z}$ (as a G_F-module), and $H^2(G_F, \mu_n^{\otimes 2})$ is identified with $H^2(G_F, \mathbb{Z}/p\mathbb{Z})$, which is the p-torsion part of the Brauer group of F. The isomorphism from $K_2(F)/pK_2(F)$ to $H^2(G_F, \mathbb{Z}/p\mathbb{Z})$ sends the class of $\ell(a)\ell(b)$ to the cyclic algebra generated by i, j subject to $i^p = a$, $j^p = b$, $ij = \xi_p ji$. I should say, also, that $H^2(G_F, \mathbb{Z}/p\mathbb{Z})$ is identified by $H^2(G_F(p), \mathbb{Z}/p\mathbb{Z})$ by the inflation map, where $G_F(p)$ is the Galois group of the maximal pro-p extension of F.

[1] $K_*(F_1)/(F_1^\times)^2$ and $K_*(F_2)/(F_2^\times)^2$ can be isomorphic as graded rings without being isomorphic as graded rings with distinguished element. $K_*(F_1)/(F_1^\times)^2$ and $K_*(F_2)/(F_2^\times)^2$ are isomorphic as graded rings iff the associated Witt-Grothendieck rings are isomorphic.

An interesting feature of the category of multifields is that products exist in certain cases. For example, if F_i, $i = 1, \dots, k$ are multifields satisfying $1 - 1 = F_i$ for each i, then the product multifield $\prod_{i=1}^k F_i$ is defined (as a set it is just $\prod_{i=1}^k F_i^\times \cup \{0\}$) and, as one might expect,

$$K_*(\prod_{i=1}^k F_i) = \prod_{i=1}^k K_*(F_i).$$

In the field case, multifield products arise rather naturally: Suppose F is a field and $S_1, \dots, S_k$ are subgroups of $F^\times$ which are open in pairwise distinct V-topologies on F and $\exists$ positive integers n_i such that $(F^\times)^{n_i} \subseteq S_i$, $i = 1, \dots, k$. Then the multifield product $\prod_{i=1}^k (F/_m S_i)$ is defined and, using the approximation theorem for V-topologies, the natural map $F/_m(\cap_{i=1}^k S_i) \to \prod_{i=1}^k F/_m S_i$ is a multifield isomorphism, so, in particular,

$$K_*(F)/(\cap_{i=1}^k S_i) \cong \prod_{i=1}^k K_*(F)/S_i.$$

The origins of this important result can be seen already in the number field case. Working in the general content of V-topologies allows one to handle orderings and valuations simultaneously.

Assume, from now on, that n is a fixed positive integer, F is a field, $\mathrm{char}(F) \nmid n$, S is a subgroup of $F^\times$ and $(F^\times)^n \subseteq S$. What can we say about the structure of $K_*(F)/S$? To what extent do orderings and valuations control this structure? There are two cases I know of where there are some answers:

(1) $F^\times/S$ is finite.
(2) S is closed under addition.

Case (2) is just the case where S is a preordering of higher level; see [B-R]. If $n = 2$ this is just a preordering in the usual sense; see [B-B] or [L]. I will concentrate here on case (1), turning briefly to case (2) at the end.

To keep things simple, let's assume further that $n = 2$, i.e., $(F^\times)^2 \subseteq S$ and $\mathrm{char}(F) \neq 2$. To get some idea of what sort of structures one might expect, suppose we have managed somehow to find subgroups $S_1, \dots, S_k$ of $F^\times$ which are open in pairwise distinct V-topologies such that $S = \cap_{i=1}^k S_i$. By our above analysis, $F/_m S \cong \prod_{i=1}^k F/_m S_i$ and $K_*(F)/S \cong \prod_{i=1}^k K_*(F)/S_i$, so we are reduced immediately to the case $k = 1$ and $S = S_1$. Our V-topology comes from an archimedean absolute value or a valuation. If it comes from an archimedean absolute value, then either S is an archimedean ordering or $S = F^\times$. Each of these cases is rather trivial. If S is an ordering (archimedean or not), then $F/_m S = \{-1, 0, 1\}$ with addition satisfying $1 + 1 = 1$, $1 - 1 = \{-1, 0, 1\}$. We denote this multifield by $\mathbb{L}_1$. (Its structure does not depend on the field and ordering we pick.) If $S = F^\times$, then $F/_m S = \{0, 1\}$ with addition satisfying $1 + 1 = \{0, 1\}$. We denote this multifield by $\mathbb{L}_0$. The associated κ-structures, $\kappa_i := K_*(\mathbb{L}_i)$, $i = 0, 1$, are easy to compute.

Suppose now that the V-topology comes from a valuation v. Typically v is not unique. Fix some such v and denote the residue field by $\overline{F}$. Suppose first that $\mathrm{char}(\overline{F}) \neq 2$. One way to ensure that S is open is to require $1 + \mathfrak{m}_v \subseteq S$. Suppose this is the case. (This assumption is not quite as far-fetched as it might appear. For example, if the valued field (F, v) is Henselian, then it is automatically the case.)

Then we have a natural short exact sequence of groups

$$0 \to \overline{F}^\times/\overline{S} \to F^\times/S \to v(F^\times)/v(S) \to 0.$$

One checks from this, using elementary valuation theory, that the structure of the multifield $F/_mS$ is completely determined by the structure of the residue multifield $\overline{F}/_m\overline{S}$ and the group $v(F^\times)/v(S)$. This implies, in turn, that the κ-structure $K_*(F)/S$ is completely determined by the κ-structure $K_*(\overline{F})/\overline{S}$ and the group $v(F^\times)/v(S)$. (It is what is called the *extension* of $K_*(\overline{F})/\overline{S}$ by the group $v(F^\times)/v(S)$.) Of course, it could be that $v(S) = v(F^\times)$, in which case $K_*(F)/S \cong K_*(\overline{F})/\overline{S}$. But, if we are lucky and this is not the case, then we can proceed by induction on $|F^\times/S|$.

The case $\mathrm{char}(\overline{F}) = 2$ is a bit different. Here, a reasonable way to ensure that S is open is to require that $1 + 4\mathfrak{m}_v \subseteq S$. Suppose, for example, that F is a finite extension of $\mathbb{Q}_2$, v is the unique extension of the 2-adic valuation, and $S = (F^\times)^2$. In this case we do have $1 + 4\mathfrak{m}_v \subseteq S$, and $F/_mS$ is one of the so-called dyadic local types $\mathbb{L}_{2k,i}$, $\mathbb{L}_{2k-1}$, $k \geq 2$, $i \in \{0,1\}$; see the survey paper [M1] for the definitions. (If $[F : \mathbb{Q}_2]$ is even, we get $\mathbb{L}_{2k,i}$ where $2k = [F : \mathbb{Q}_2] + 2$ and

$$i = \begin{cases} 0 \text{ if } \sqrt{-1} \in F^\times \\ 1 \text{ if } \sqrt{-1} \notin F^\times \end{cases}.$$

If $[F : \mathbb{Q}_2]$ is odd, we get $\mathbb{L}_{2k-1}$ where $2k - 1 = [F : \mathbb{Q}_2] + 2$. To understand what is going on here one needs to know some local classfield theory.) The associated κ-structures are denoted by $\kappa_{2k,i}$, κ_{2k-1}, $k \geq 2$, $i \in \{0,1\}$.

Thus we see that, modulo various unwarranted assumptions, $K_*(F)/S$ is either κ_0 or it is a finite product of (one or more) κ-structures, each of which is either κ_1, an extension of something simpler, or one of the dyadic local types $\kappa_{2k,i}$, κ_{2k-1}, $k \geq 2$, $i \in \{0,1\}$.[2] The class of elementary types is the smallest class of κ-structures containing κ_i, $i \in \{0,1\}$, $\kappa_{1,1}$ and the dyadic local types $\kappa_{2k,i}$, κ_{2k-1}, $k \geq 2$, $i \in \{0,1\}$ and closed under formation of finite products and extensions by a finite group of exponent 2. It is known that if $K_*(F)/S$ is of elementary type, then so is $K_*(F)/S'$, for any $S' \supseteq S$. The elementary type conjecture (at least, a version of it) is that $K_*(F)/(F^\times)^2$ is of elementary type, for any field F, $\mathrm{char}(F) \neq 2$, with $|F^\times/(F^\times)^2| < \infty$. Various people have worked on this; see the survey paper [M1]. On the positive side, it is known that every elementary type is realized by a field, i.e., has the form $K_*(F)/S$ for some field F, $\mathrm{char}(F) \neq 2$ and some subgroup S of $F^\times$. It is even possible to arrange things so that $S = (F^\times)^2$, if we want. The construction of the field F involves lots of valuation theory; see [Ku]. Also, the structure of elementary types is well-understood. For example, given $d \geq 0$, one can count the number of non-isomorphic elementary types $K_*(F)/S$ with $|F^\times/S| = 2^d$. Actually, we get two numbers $e(d)$, $\overline{e}(d)$ here, depending on whether we view the κ-structures as graded rings with distinguished element or just as graded rings.

[2]Actually, it is necessary to include an additional building block, $\kappa_{1,1} = K_*(\mathbb{L}_{1,1})$, where $\mathbb{L}_{1,1} = \{-1, 0, 1\}$ with addition satisfying $1 + 1 = \{-1, 1\}$, $1 - 1 = \{-1, 0, 1\}$. This is the same as the extension of κ_0 by a cyclic group of order 2, but the distinguished element is not the same.

Both numbers are useful; e.g., see [Mi-S]. Here are the first few values:

d	$e(d)$	$\overline{e}(d)$
0	1	1
1	3	2
2	6	5
3	17	15
4	51	47
5	155	145
6	492	470

There has also been work on the related problem of detecting valuations from their K-theoretic 'footprints'. (E.g., if $K_*(F)/S$ decomposes somehow, as an extension or product, is it possible to find valuations witnessing this fact?) Bröcker's trivialization theorem for fans [B2] is an early result of this type. More recent results of this sort are found in [A-E-J] and [J-W2]. These results are applied, in [J-W1] and [J-W2], to understand the structure of the Galois group $G_F(2)$, for $K_*(F)/(F^\times)^2$ of elementary type. (Roughly, products correspond to free products of Galois groups, extensions correspond to certain semi-direct products of Galois groups, and dyadic local types correspond to Demushkin pro-2 groups.) More recently still, beginning with the papers [E], [H-J] and [Ko], the conjecture, as well as parts of the theory, have been extended to the case $n = p$, p an odd prime (assuming $\mu_p \subseteq F^\times$). I should say that, for $n = p$, p prime, the elementary type conjecture is known to be true for $|F^\times/(F^\times)^p| = p^d$, where

$$\begin{cases} d \leq 5 \text{ if } p = 2 \\ d \leq 4 \text{ if } p \neq 2 \end{cases} .$$

In case (2) one has satisfactory answers, valid for any even n, but they are phrased in terms of the higher level reduced Witt ring $W_S(F)$ rather than in terms of the relative Milnor ring $K_*(F)/S$. What one gets is certain local-global principles, relating the structure of $W_S(F)$ to the structure of the various quotients $W_{S'}(F)$, where S' is a preordering of higher level, $S \subseteq S'$ and $|F^\times/S'| < \infty$. In particular, one has a local-global principle for reduced isotropy and a representation theorem for the reduced Witt ring. There are also valuation-theoretic formulations of these results. See [B-B] for the case $n = 2$, [B-R] for the general case. See [B1] and [Pr] for earlier formulations of the local-global principle for reduced isotropy. It is not clear, to me at least, how one goes about translating these results about $W_S(F)$ into results about $K_*(F)/S$, at least if $n > 2$. Despite this, it seems that the arguments in [Po] do imply, for subgroups S of $F^\times$ satisfying *both* (1) and (2), $K_*(F)/S$ is of elementary type (in a suitably generalized sense), for any even n. For $n = 2$ this was known earlier; see [C] or [Ku].

References

[A-E-J] J. Arason, R. Elman, B. Jacob, *Rigid elements, valuations, and realization of Witt rings*, J. Algebra 110 (1987), 449–467. MR0910395 (89a:11041)

[B-B] E. Becker, L. Bröcker, *On the description of the reduced Witt ring*, J. Algebra 52 (1978), 328–346. MR0506029 (58:21935)

[B-R] E. Becker, A. Rosenberg, *Reduced forms and reduced Witt rings of higher level*, J. Algebra 92 (1985), 477-503. MR0778463 (86e:11029)

[B1] L. Bröcker, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math. Ann. 210 (1974), 233–256. MR0354549 (50:7027)

[B2] ———, *Characterization of fans and hereditarily Pythagorean fields*, Math. Z. 151 (1976), 149–163. MR0422233 (54:10224)

[C] T. Craven, *Characterizing reduced Witt rings of fields*, J. Algebra 53 (1978), 74–96. MR0480332 (58:505)

[E] I. Efrat, *Orderings, valuations and free products of Galois groups*, Sém. Struct. Alg. Ordonnées 54, Univ. Paris 7 (1995).

[H-J] Y.S. Hwang, B. Jacob, *Brauer group analogues of results relating the Witt ring to valuations and Galois theory*, Canad. J. Math. 47 (1995), 527–543. MR1346152 (97a:12004)

[J-W1] B. Jacob, R. Ware, *A recursive description of the maximal pro-2 Galois group via Witt rings*, Math. Z. 200 (1989), 379–396. MR0978598 (90b:11127)

[J-W2] ———, *Realizing dyadic factors of elementary type Witt rings and pro-2 Galois groups*, Math. Z. 208 (1991), 193–208. MR1128705 (92h:11032)

[Ko] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterization of p-adic fields)*, J. Reine Angew. Math. 465 (1995), 165–182. MR1344135 (96m:12003)

[Ku] M. Kula, *Fields with prescribed quadratic form schemes*, Math. Z. 167 (1979), 201–212. MR0539104 (80f:10024)

[L] T.-Y. Lam, *Orderings, valuations and quadratic forms*, CBMS 52, Amer. Math. Soc., 1983. MR0714331 (85e:11024)

[M1] M. Marshall, *The elementary type conjecture in quadratic form theory*, Cont. Math. 344 (2004), 275–293. MR2060204 (2005b:11046)

[M2] ———, *Real reduced multirings and multifields*, J. Pure and Applied Algebra 205 (2006), 452–468. MR2203627 (2006k:14110)

[M-S] A.S. Merkurjev, A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 1011–1046 (Russian); Math. USSR Izv. 21 (1983), 307–340 (English translation). MR0675529 (84i:12007)

[Mi] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. 9 (1970), 318–344. MR0260844 (41:5465)

[Mi-S] J. Mináč, T. Smith, *Decomposition of Witt rings and Galois groups*, Canad. J. Math. 47 (1995), 1274–1289. MR1370518 (96k:11048)

[Pf] A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. (1966), 116–132. MR0200270 (34:169)

[Po] V. Powers, *Characterizing reduced Witt rings of higher level*, Pac. J. Math. 128 (1987), 333–347. MR0888522 (88d:11038)

[Pr] A. Prestel, *Lectures on formally real fields*, IMPA Lecture Notes 22, Rio de Janeiro, 1975, Lecture Notes in Math. 1093, Springer, 1984. MR0769847 (86h:12013)

[V1] V. Voevodsky, *Motivic cohomology with $\mathbb{Z}/2$-coefficients*, Publ. Math. IHES 98 (2003), 59–104. MR2031199 (2005b:14038b)

[V2] ———, *On motivic cohomology with $\mathbb{Z}/\ell$-coefficients*, preprint.

[W] E. Witt, *Theorie der Quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. 176 (1937), 31–44.

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