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Introduction to quadratic forms over fields, by T. Y. Lam, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005, xxi+550 pp., ISBN 978-0-8218-1095-8

The arithmetic theory of quadratic forms goes back to the earliest days of mathematics: for instance, sums of squares of integers or rational numbers were already mentioned in the work of Diophantus. Quadratic forms over algebraic number fields and their rings of integers became prominent in the development of class field theory in the early 20th century.

By contrast, quadratic forms over *arbitrary fields* were rarely—if at all—considered before 1937. This date is very important in the so-called *algebraic theory of quadratic forms*, because this is the year of the publication of the pioneering paper by Ernst Witt [26], "Quadratische Formen über beliebigen Körpern" (Quadratic forms over arbitrary fields). This paper is innovative in many ways. As already mentioned, even the idea of taking coefficients in an arbitrary field rather than a "natural" field, such as for instance an algebraic number field, was a new one. Moreover, whereas all the previous authors considered quadratic forms individually, Witt introduced the new concept of studying them collectively. This notion, called the *Grothendieck–Witt ring* of a field, is built up from the isomorphism classes of nondegenerate quadratic forms in the same way that ring of integers is constructed from natural numbers. From the Grothendieck–Witt ring we then obtain the *Witt ring* of the field, in which the "trivial" quadratic forms, that is the *hyperbolic ones* (nondegenerate quadratic forms having the largest possible isotropic subspace) are identified to the zero element. One of the main results of this paper is the so-called *Witt cancellation theorem*, namely that stably isomorphic quadratic forms are isomorphic. Using this, one shows that the class of a quadratic form in the Grothendieck–Witt ring determines the form. Moreover, two nondegenerate quadratic forms of the same dimension are isomorphic if and only if they have the same image in the Witt ring. Hence the classification problem of quadratic forms is essentially equivalent to the determination of the Witt ring.

Early breakthroughs and Lam's first book

With Witt's 1937 paper a new topic was born, the theory of quadratic forms over fields. Even though the paper did not go unnoticed, very little progress was made until the late 1960's. At that time, a series of papers by Pfister changed this situation—not only did he prove beautiful new results, but the notions and methods he introduced inspired many other mathematicians to work on the algebraic theory of quadratic forms. Among those was T. Y. Lam, who, in 1973, published the first textbook [10] dealing with this topic. This book was an immediate success. It was very timely, appearing only a short time after Pfister's papers, and also contained important results of Knebusch and Scharlau. Moreover, the book is extremely well written and accessible even for beginning students—the existence of this book

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was an important contribution to the quick development of the algebraic theory of quadratic forms in the 1970's and beyond.

Let us go back to the late 1960's to survey Pfister's work. Let F be a field of characteristic $\neq 2$. Many long-standing questions concerned sums of squares of elements of F: for instance, the *level* of the field F, denoted by $s(F)$, is by definition the smallest positive integer s such that -1 can be written as a sum of s squares (we say that $s(F)$ is infinite if no such s exists). Is $s(F)$ always a power of 2? This was conjectured by Kaplansky in 1953, and it was generally believed to be true, but no proof was available before Pfister. In 1968, using the theory of multiplicative quadratic forms, today called *Pfister forms*, he proved in a beautiful and elegant paper that indeed $s(F)$ is either infinite or a power of 2. Another consequence of his theory is that the product of two sums of 2^n squares is also a sum of 2^n squares, generalizing the well-known results for sums of 2, 4 and 8 squares.

Ever since their invention in the late 1960's, Pfister forms continue to play a basic role in the algebraic theory of quadratic forms. They are very easy to define: if $a_1, \dots, a_n$ are nonzero elements of F, we denote by $\langle a_1, \dots, a_n\rangle$ the quadratic form $a_1X_1^2 + \cdots + a_nX_n^2$. The associated n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is by definition the tensor product of the n binary forms $\langle 1, a_i\rangle$. This is a quadratic form in 2^n variables; the special case where all the a_i's are equal to 1 corresponds to sums of 2^n squares. Let us denote by $W(F)$ the Witt ring of F, and let $I(F)$ be the ideal of Witt classes of even-dimensional forms. It turns out that the powers $I^n(F)$ play an essential role in the study of the Witt ring—and it is easy to see that they are generated by the n-fold Pfister forms. The famous Arason–Pfister theorem, proved in 1970, states that the intersection of the $I^n(F)$'s ($n \geq 1$) is trivial.

When are two quadratic forms isomorphic? Milnor's conjecture and Voevodsky's theorem

The paper of Milnor, "Algebraic K-theory and quadratic forms" [12], is another one of the landmarks of the theory. It also appeared in 1970, and it contains a famous conjecture that inspired many authors in the late 20th and early 21st century. The starting point is the classification problem of quadratic forms via cohomological invariants. Indeed, the simplest invariants of quadratic forms are the dimension (or number of variables) and the determinant. If we want invariants defined on $W(F)$, then we have to consider dimension mod 2—indeed, the invariant must vanish on hyperbolic forms, and these may assume any even dimension. Hence, dimension mod 2 gives us a map $e_0 : W(F) \to \mathbf{Z}/2\mathbf{Z}$, the kernel of which is by definition $I(F)$; thus we obtain an isomorphism $e_0 : W(F)/I(F) \simeq \mathbf{Z}/2\mathbf{Z}$. Similarly, the *determinant* induces a homomorphism $e_1 : I(F)/I^2(F) \to F^*/F^{*2}$, and it is easy to see that e_1 is an isomorphism. The last of the classical invariants, the *Hasse–Witt invariant*, also provides a homomorphism $e_2 : I^2(F)/I^3(F) \to \mathrm{Br}_2(F)$. It is a basic remark that the target groups can all be identified to Galois cohomology groups with mod 2 coefficients. Indeed, let F_s be a separable closure of F, and let $\Gamma_F = \mathrm{Gal}(F_s/F)$. For all positive integers n, set $H^n(F) = H^n(\Gamma_F, \mathbf{Z}/2\mathbf{Z})$. Then $H^0(F) = \mathbf{Z}/2\mathbf{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$. For all $a \in F^*$, let us denote by (a) the image of a in $H^1(F)$. In his 1970 paper, Milnor conjectured that for every positive integer n, there exists a well-defined isomomorphism

$$e_n : I^n(F)/I^{n+1}(F) \to H^n(F)$$

sending the n-fold Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ to the cup product $(-a_1) \cup \cdots \cup (-a_n)$.

The only easy cases of Milnor's conjecture are $n = 0$ and $n = 1$. Even the case $n = 2$ is very difficult, and was only proved in 1981 by Merkurjev using methods from algebraic K-theory and algebraic geometry. Even though later on Arason and Wadsworth found elementary proofs for this case, this seems to be hopeless for higher values of n. The case $n = 3$ was settled by Merkurjev and Suslin, and independently by Rost in the early 1990's. Their proofs are highly nontrivial and nonelementary, relying on methods of algebraic K-theory. Rost also announced the proof of the case $n = 4$, but the general case still seemed out of reach at that time. However, in the following few years, Voevodsky developed new tools, in particular motivic cohomology, that made it possible to prove Milnor's conjecture (cf. [14], [23], [24], [13], see also the surveys [7], [16]). In 2002 Voevodsky received the Fields Medal for this work.

Milnor's conjecture, today Voevodsky's theorem, provides a classification of quadratic forms via "secondary invariants". It is a beautiful and important theorem, but in a way this is not really what we want! Is it possible to find a complete set of invariants defined on $W(F)$ itself? This is not known so far.

When does a quadratic form have a nontrivial zero?

Many important results and open questions in the algebraic theory of quadratic forms concern the notion of isotropy. We say that a quadratic form $q : V \to F$ is *isotropic* if there exists a nonzero $x \in V$ such that $q(x) = 0$, and it is *anisotropic* otherwise. The maximal dimension of an anisotropic quadratic form over F is an important invariant of F, called the *u-invariant* (for lack of a better name!), and it is denoted by $u(F)$. This invariant is somewhat reminiscent of the level of F—indeed, $s(F)$ is the highest dimension of an anisotropic unit form $\langle 1, \dots, 1\rangle$ over F. In 1953, Kaplansky conjectured that $u(F)$ is either infinite or a power of 2—and this conjecture was believed plausible by many. It is easy to prove that there exist fields of u-invariants 1, 2 and 4, and more generally any power of 2, and that no field can have u-invariant 3, 5 or 7. Therefore, it came as a big surprise when Merkurjev proved in 1989 the existence of a field with a u-invariant equal to 6, and then, in 1990, that for any even number n there exists a field with u-invariant n ! The method used by Merkurjev, called *index reduction*, uses the theory of central simple algebras, and also relies on algebraic geometric techniques. More recently, Izhboldin [5] and Vishik [22] proved that *odd integers* also occur as u-invariants: Izhboldin constructed fields with u-invariant 9, and Vishik fields with u-invariant $2^n + 1$ for any $n \geq 3$. The method of Vishik uses tools from algebraic geometry. Even though Kaplansky's conjecture does not hold in general, it is worth noting that the fields constructed by Merkurjev, Izboldhin and Vishik are huge, and that the conjecture might be true for fields of finite type over $\mathbf{Q}$.

Another topic related to isotropy is the theory of *generic splitting* introduced by Knebusch. Let q be a quadratic form, and let $F(q)$ be the function field of the associated conic. Then clearly q becomes isotropic over $F(q)$. Taking the anisotropic part of this form and continuing this process, we get a sequence of quadratic forms over bigger and bigger fields, and in particular a sequence of positive integers given by the dimensions of these forms. This is called the *splitting pattern* of the quadratic form. The determination of these splitting patterns is important for the understanding of isotropy properties of quadratic forms. Recently, many important results were obtained on this topic, in particular by Hoffmann, Karpenko and Vishik. For instance, Karpenko proved the following statement, conjectured by Hoffmann: if q is an anisotropic quadratic form belonging to $I^n(F)$ such that $\dim(q) < 2^{n+1}$, then $\dim(q)$ is of the form $2^{n+1} - 2^i$ for some integer $i \in \{0, \dots, n\}$. Once more, the methods are based on algebraic geometry, in particular the study of Chow groups of certain algebraic varieties related to the quadratic form.

From quadratic forms to linear algebraic groups

Since the early 1960's, it was clear that the study of quadratic forms can be placed in the more general context of noncommutative Galois cohomology of linear algebraic groups, in particular classical groups. Indeed, let q be a quadratic form and let us consider the group of all its isometries, called the *orthogonal group* $O(q)$ of the quadratic form q. Then $O(q)$ is a linear algebraic group defined over F. The set of all isomorphism classes of quadratic forms q' defined over q such that q' and q become isomorphic over the separable closure F_s can be identified with the Galois cohomology set $H^1(\Gamma_F, O(q)(F_s))$. We denote this set by $H^1(F, O_q)$. One can define $H^1(F, G)$ for any linear algebraic group G over F. It is natural to reformulate the classical results concerning quadratic forms in this framework, and to ask whether they can be generalized to other linear algebraic groups. For instance, a theorem of Springer proved in 1951 states that if two quadratic forms become isomorphic over an odd degree extension, then they are isomorphic over the ground field. This is equivalent to saying that if L is an odd degree extension of F, then the canonical map of pointed sets $H^1(F, O_q) \to H^1(L, O_q)$ is injective. The question of generalizing this, and other well-known results concerning quadratic forms, was raised by Serre (see [18], [19]). This provided inspiration for intensive research (see for instance [21], [1], [2], [3], [4], and the survey [20]). In case G is a classical group, a basic tool is provided by the result of André Weil [25] stating that G can be obtained from an algebra with involution. Algebras with involution, and the closely related notion of hermitian forms, are studied extensively in the books of Knus, Merkurjev, Rost and Tignol (*The Book of Involutions*) [9], Scharlau [17], and Knus [8]. Several of the questions of Serre's 1962 paper are now (at least partially) solved, but there are still many open problems—old and new !

Lam's second book

Since 1973, the specialists in the field of quadratic forms grew up reading Lam's book [10]. It contains all the basic facts explained in a wonderfully clear way. The book was justly rewarded with the Leroy P. Steele prize in Mathematical Exposition in 1982. A victim of its success, it went out of print by the late 1970's; the second printing with revisions issued in 1980 was not sufficient for the demand, and also went out of print very quickly. Today's students also need this excellent resource, both for learning about quadratic forms over fields and for taking part in working on the many open problems outlined above.

In view of all the recent progress, Lam decided to write a new book, based on the first one, but with many chapters rewritten and two chapters added with new material. The first 11 chapters are revised and augmented versions of the corresponding chapters of [10]. Chapter 12 is entitled "Special topics in Quadratic Forms". It contains a selection of results, for instance, isomorphisms of Witt rings, quadratic forms of low dimension, behavior of Witt rings under biquadratic extensions, and several others. Chapter 13 is a complement of Chapter 11, and concerns field invariants. In particular, Merkurjev's construction of fields of u-invariant 6 is presented here. Many new results are given with full proofs, others are only quoted. There are many interesting exercises, and also a list of challenging open problems. The title has also changed, it is now "Introduction to quadratic forms over fields". Of course it still has the same lively, inspiring and very clear style of its predecessor.

The book is readable with only a basic previous knowledge of algebra. It is remarkable how much Lam is able to teach his reader with so few prerequisites. The new book has more than doubled in size, and contains many interesting results explained in an elementary way. Of course, it was not possible to present in detail some of the recent progress outlined in this survey, because more sophisticated notions and methods would have been necessary, such as Galois cohomology and Chow groups. Still, Lam's book is the right place to start, an important first step before reading research articles and some of the other books, such as *The Book of Involutions* [9]. It belongs on the shelves of every mathematical library, and is an excellent choice of a textbook for a course on quadratic forms over fields. In every aspect, *Introduction to Quadratic Forms Over Fields*" is a great book, invaluable both for learning the topic and as reference.

References

[1] E. Bayer-Fluckiger, H.W. Lenstra, Jr., Forms in odd degree extensions and self-dual normal bases, *Amer. J. Math.* **112** (1990), 359-373. MR1055648 (91h:11030)

[2] E. Bayer-Fluckiger, R. Parimala, Galois cohomology of linear algebraic groups over fields of cohomological dimension ≤ 2, *Invent. Math.*, **122** (1995) MR1358975 (96i:11042)

[3] J.-L. Colliot-Thélène, Ph. Gille, R. Parimala, Arithmetic of linear algebraic groups over two-dimensional geometric fields, *Duke Math. J.* **121** (2004), 285-321. 195–229. MR2034644 (2005f:11063)

[4] Ph. Gille, Cohomologie galoisienne des groupes quasi-déployes sur des corps de dimension cohomologique ≤ 2, *Comp. Math.* **125** (2001), 283-325. MR1818983 (2002c:11045)

[5] O. Izhboldin, Fields of u-invariant 9, *Ann. of Math.* **154** (2001), 529–587. MR1884616 (2002m:11026)

[6] B. Kahn, La conjecture de Milnor, d'après Voevodsky, *Séminaire Bourbaki* (1996/97), Exp. **834** (juin 1997), *Astérisque* **245** (1997), 379–418. MR1627119 (2000a:19002)

[7] B. Kahn, Formes quadratiques et cycles algébriques, d'après Rost, Voevodsky, Visik, Karpenko, *Séminaire Bourbaki* (2004/05), Exp. **941** (novembre 2004), *Astérisque* **245** (1997), 379–418. MR2296417

[8] M. Knus, *Quadratic and hermitian forms over rings*, Grundlehren der Math. Wiss. **294** Springer-Verlag (1991). MR1096299 (92i:11039)

[9] M. Knus, A. Merkurjev, M. Rost, J-P. Tignol, *The Book of Involutions*, AMS Colloquium Publications, Vol 44 (1998).

[10] T.Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin (1973) (revised printing 1980). MR0396410 (53:277)

[11] A.S. Merkurjev, Kaplansky's conjecture in the theory of quadratic forms (in Russian), *Zap. Nauchn. Sem. Leningrad* Otdel. Mat. Inst. Steklov (LOMI) **175** (1989), Kolsta i Moduli **3**, 75–89, 163–164 (English translation : *J. Soviet Math.* **57** (1991), 3489–3497). MR1047239 (90m:11058)

[12] J. Milnor, Algebraic K-theory and quadratic forms, *Invent. Math.* **9** (1970), 318–344. MR0260844 (41:5465)

[13] F. Morel, Milnor's conjecture on quadratic forms and mod 2 motivic complexes, *Rend. Sem. Mat. Univ. Padova*, **114** (2005), 63-101. MR2207862 (2006m:14027)

[14] D. Orlov, A. Vishik, V. Voedovsky, An exact sequence for Milnor's K-theory with applications to quadratic forms, preprint (2001), arxiv.org/abs/math/0101023

[15] A. Pfister, *Quadratic forms with applications to algebraic geometry and topology*, LMS Lecture Note Series **217**, Cambridge University Press (1995). MR1366652 (97c:11046)

[16] A. Pfister, On the Milnor Conjectures, History, Influence, Applications, *Jahresbericht DMV* **102** (2000), 15–41. MR1769021
[17] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der Math. Wiss. **270** Springer-Verlag (1985). MR770063 (86k:11022)
[18] J-P. Serre, Cohomologie galoisienne des groupes algébriques linéaires, *Colloque de Bruxelles*, 1962, 53–67 (=Collected Papers, 53). MR0186719 (32:4177)
[19] J-P. Serre, *Cohomologie galoisienne*, Cinquième édition, Lecture Notes in Mathematics 5, Springer-Verlag (1964, 1994) MR1324577 (96b:12010)
[20] J-P. Serre, Cohomologie galoisienne : progrès et problèmes, *Séminaire Bourbaki* 1993–1994, exposé 783. MR1324577 (96b:12010)
[21] R. Steinberg, Regular elements of semisimple algebraic groups, *Pulb. Math. IHES* **25** (1965), 49–80. MR0180554 (31:4788)
[22] A. Vishik, Fields of u-invariant $2^r + 1$, preprint (2006).
[23] V. Voevodsky, Reduced power operations in motivic cohomology, *Publ. Math. IHES* **98** (2003), 1–57. MR2031198 (2005b:14038a)
[24] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2\mathbf{Z}$-coefficients, *Publ. Math. IHES* **98** (2003), 59–104. MR2031199 (2005b:14038b)
[25] A. Weil, Algebras with involutions and the classical groups, *J. Indian Math. Soc.* (N.S.) **24** (1960) (= Collected Papers, 1960 b). MR0136682 (25:147)
[26] E. Witt, Theorie der quadratischen Formen in beliebigen Körpern, *J. Reine Angew. Math.* **176** (1937), 31–44 (= Collected Papers, 1).

Eva Bayer-Fluckiger
Ecole Polytechnique Fédérale, Lausanne
E-mail address: eva.baver@epfl.ch