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Moonshine beyond the monster: The bridge connecting algebra, modular forms and physics, by Terry Gannon, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, Massachusetts, 2006, 492 pp., US$130.00, ISBN 978-0-521-83531-2

In about 1972 R. L. Griess and Fischer independently suggested the existence of a new sporadic simple group, which had a double cover of the "baby monster" discovered by Fischer as the centralizer of an involution. Calculations suggested that its smallest non-trivial complex representation probably had dimension 196883. "Moonshine" is the attempt to explain John McKay's extraordinary observation that this number is almost equal to the coefficient 196884 of the elliptic modular function

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

giving the j-invariant of the elliptic curve $\mathbb{C}/\{1,\tau\}$ (where $q = e^{2\pi i\tau}$). At first, a common explanation was that there are many large numbers that turn up in mathematics, and a few of them will be almost equal just by coincidence.

Shortly afterwards, John Thompson [T] pointed out that the next coefficient 21493760 of the elliptic modular function is equal to the sum of the dimensions of the first 3 irreducible representations of the monster, and similarly all the other coefficients seems to be simple linear combinations of dimensions of irreducible representations. He conjectured that this might be because the monster acts on a graded representation V whose piece of degree n is the coefficient of q^n in $j(\tau) - 744$, and suggested looking at the traces of other elements of the monster on this representation. (The coefficient 744 of q^0 is a historical accident as adding an arbitrary constant to j still gives a function with similar properties. The most natural normalization is to set the constant term equal to 24, the number given by Rademacher's infinite series for coefficients of the j function.)

Conway and Norton [CN] carried out Thompson's suggestion and found that the traces of all elements of the monster seem to be given by the coefficients of Hauptmoduls (roughly, functions invariant under a genus 0 congruence subgroup of $SL_2(\mathbb{R})$). Atkin, Fong, and Smith [S] verified that the monster indeed has a representation with these properties by checking that the Hauptmoduls satisfied the necessary congruence and positivity conditions.

The monster simple group was finally constructed by R. L. Griess [G] in 1982. Norton had observed a few years before that the monster probably had a commutative (but non-associative) algebra structure on its 196883 dimensional representation, and it was also known how to write the 196884 dimensional representation as a sum of three irreducible representations of a centralizer of an involution. The monster could therefore be constructed by adjusting the finite number of parameters that an algebra structure depends upon so that the algebra has an extra automorphism. While the idea is simple, the details are horrendous to carry out: John Conway told me that when he heard Griess had constructed the monster, he assumed that Griess must have found another way to do it, because he could not believe that anyone would be patient enough to finish this calculation.

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Victor Kac [K] pointed out that the Lie group E_8 has a similar natural infinite dimensional representation, obtained by restricting an irreducible representation of the affine Lie algebra E_8, except that the traces of elements need not be Hauptmoduls but can be more general modular functions. More generally he showed that something similar happens for any highest weight representation of any affine Kac-Moody algebra when restricted to a suitable compact Lie group. Although the monster corresponds to a compact Lie group, no analogue of the corresponding affine Lie algebra is known for it.

Frenkel, Lepowsky, and Meurman [FLM] constructed an infinite dimensional module for the monster. This has the structure of a vertex algebra, and contains the Griess algebra inside it: one of the vertex algebra products is the Griess product. Some of the infinite dimensional representations of affine Lie algebras also carry similar vertex algebra structures, again showing that moonshine for the monster is very closely related to Kac's theory of affine Lie algebras.

The representation constructed by Frenkel, Lepowsky, and Meurman does indeed satisfy the moonshine conjectures; the proof of this uses a "generalized Kac-Moody algebra" acted on by the monster that is constructed using bosonic string theory. There is a small extension of Kac-Moody algebras called generalized Kac-Moody algebras, which unlike Kac-Moody algebras are allowed to have imaginary simple roots. Their theory is very similar: the basic theorems about them were proved by copying out Kac's proofs for the case of Kac-Moody algebras and making the obvious minor changes (though Kac pointed out to me that I introduced some completely original errors). The monster group does not exactly correspond to a Kac-Moody algebra; instead it acts as a group of diagram automorphisms of a generalized Kac-Moody algebra called the monster Lie algebra. The action of the monster group on the monster Lie algebra is similar to the "triality" action of the cyclic group of order 3 acting on the Lie algebra of the orthogonal group in 8 dimensions.

The proof of the moonshine conjecture does not directly show that the McKay-Thompson functions are genus 0. Instead it shows that they are completely replicable; this means that they satisfy identities similar to the product formula for the j-function. At first the fact that the completely replicable functions were Hauptmoduls was verified case by case, but more recently Cummins and Gannon [CG] found a more satisfactory uniform proof of this.

Some sporadic groups in the monster have natural graded representations that have vertex algebra structures only if they are reduced modulo some prime. For example, the baby monster has a representation on a graded space with character

$$q^{-1} + 4372q + 96256q^2 + \cdots$$

which cannot have a natural vertex algebra structure as the piece of dimension 4372 has no non-trivial product invariant under the baby monster. However Ryba [Ry] discovered that it probably does if reduced modulo 2, and in this case the whole space (probably) has a vertex algebra structure over the field with 2 elements invariant under the baby monster. Many, and possibly all, of the finite groups of Lie type also act naturally on vertex algebras over finite fields, which suggests the possibility that all finite simple groups act on suitable vertex algebras. However, for the sporadic groups not involved in the monster, no one has found any evidence of this, except possibly in the case of the Rudvalis group.

Conway and Norton [CNS] also found a particularly elegant presentation for the monster (or rather a closely related group) as a Coxeter group together with one simple extra relation. The fact that this is indeed a presentation for the monster was proved by A. A. Ivanov [I]. The relation between this presentation and the moonshine properties of the monster is unclear, though Miyamoto [M] has made some progress by showing that the generators of the presentation have a reasonably natural description as automorphisms of the monster vertex algebra; his construction uses the Niemeier lattice E_6^4, which makes one wonder if there are similar constructions for the other Niemeier lattices.

Another of McKay's strange observations about E_8 is that orbits of pairs of involutions of type 2A in the monster seem to correspond to the 9 nodes of the affine E_8 diagram. (There is a similar relation between the baby monster and the E_7 diagram, and between the sporadic group Fi_{24} and the E_6 diagram, so this is probably not a coincidence.) Recently Lam and Miyamoto [LM] found certain subalgebras of the monster vertex algebra corresponding to these nodes, which seems to go a long way towards explaining McKay's observation.

The proof of the moonshine conjectures depends on several coincidences. Even the existence of the monster seems to be a fluke in any of the known constructions: these all depend on long, strange calculations that just happen to work for no obvious reason, and would not have been done if the monster had not already been suspected to exist. Then the dimension of the Leech lattice just happens to be just 2 less than the critical dimension 26 of string theory, which is just what is needed for the no-ghost theorem to be used to construct the monster Lie algebra. The monster Lie algebra just happens to have a Weyl vector, which is extremely unusual for algebras constructed like this, and means that its simple roots can be described explicitly.

A major problem with studying the monster vertex algebra is that it is constructed by splicing together two quite different pieces. By analogy, the E_8 is also sometimes constructed as a sum of two pieces: a subalgebra D_8, and a half spin representation S of D_8. The Lie bracket on D_8 is already given, and the bracket of D_8 with S is given by the action of D_8 on S, and the bracket of two elements of S can be given by adjointness, using the fact that S has an invariant bilinear form. The resulting bracket on $D_8 \oplus S$ just happens to satisfy the Jacobi identity, so we get a Lie algebra E_8. The construction of the monster vertex algebra is formally very similar. In the case of E_8 there is a more natural construction which writes the E_8 Lie algebra as just one piece, rather than the sum of two different pieces. For this, one just takes the vertex algebra of (the double cover of) the E_8 lattice; the degree 1 piece of this is the E_8 Lie algebra. (It is not hard to eliminate vertex algebras from this construction and get a completely elementary construction of the E_8 Lie algebra as "one piece".) A major open problem is to find an analogous natural construction of the monster vertex algebra as just one piece, rather than as the sum of several unrelated smaller pieces. Duncan recently managed to do this for a similar superalgebra acted on by the double cover of Conway's simple group: he showed that the vertex superalgebra was isomorphic to the superalgebra of a certain 12 dimensional lattice. Later in [D], he used a similar idea to find a vertex algebra acted on by the Rudvalis groups: this is the first indication that it might be possible to extend moonshine to some of the 6 sporadic groups not involved in the monster.

If a finite group acts on a vertex algebra, then the corresponding "orbifold" vertex algebra is easy to define: it is just the subalgebra fixed by the finite group. However it seems very hard to describe the representations of this subalgebra and the fusion between them, though there has been considerable progress by Dong, Li and Mason [DLM]. If this problem could be solved, it would quickly lead to a solution of the generalized moonshine conjectures, as it would be possible to construct analogues of the monster Lie algebra for all elements of the monster. One way of approaching orbifolds is via nets of von Neumann algebras, which seem to be closely related to vertex algebras: although there seems to be no natural functor going in either direction, whenever there is a construction of an interesting vertex algebra, there seems to be a similar construction of a net of von Neumann algebras, and vice versa. For example, Kawahigashi and Longo [KL] recently gave a construction of a net of von Neumann algebras with automorphism group the monster, by following the usual construction of the monster vertex algebra. For nets of von Neumann algebras, there is a good theory of orbifolds. Unfortunately it is not obvious how to use this to construct the Lie algebras associated to elements of the monster: the problem is that the elements of these Lie algebras correspond to unbounded operators, which are not so easy to get hold of using von Neumann algebras of bounded operators. So vertex algebras see the unbounded operators but cannot handle orbifolds well, while von Neumann algebras can handle orbifolds but cannot see the unbounded operators. Perhaps there is some way to combine vertex algebras and nets that can handle both orbifolds and unbounded operators, and therefore prove the generalized moonshine conjectures.

Hirzeburch [H] has suggested a "prize question" about the existence of a 24 dimensional manifold acted on by the monster, whose cohomology of twisted Dirac operators would give the monster vertex algebra. Hopkins and Mahowald [HM] showed the existence of a manifold with the topological properties suggested by Hirzebruch, though it does not seem to be known whether this manifold has a suitable action of the monster.

The most recent new idea about the monster is Witten's suggestion [W] of a relation between the monster and three dimensional gravity, which I am still too baffled by to say anything useful about.

At the moment we can prove almost anything we want about the monster and moonshine (or could with a bit more effort) but are really short of good explanations for what is going on. For example, there is still no expanation of why the monster exists that does not involve many pages of obscure calculations.

The book under review is a survey of some areas of mathematics related to moonshine. It is written in a lively and readable style, and the author explains what is going on instead of giving formal proofs. Much of the book is only indirectly related to the monster group: for example, there is a long chapter on conformal field theory, and an explanation of the relation between subfactors, knot polynomials, and quantum groups. It is an excellent introduction to this area for anyone who is looking for an informal survey.

One aspect of moonshine that is not discussed much in the present book is the relation between generalized Kac-Moody algebras of rank greater than 2 and automorphic forms of high rank. This is covered in a recent book [Ra] by U. Ray.

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