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SELECTED MATHEMATICAL REVIEWS

related to papers in the previous section and this section by
FORRESTER & WARNAAR AND BAAS & SKAU

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Erdös, P.

On a new method in elementary number theory which leads to an elementary proof of the prime number theorem.

Proc. Nat. Acad. Sci. U. S. A. **35**, (1949), 374–384

The prime number theorem (PNT) asserts that

$$\pi(x) \sim x/\log x, \quad x \to \infty,$$

where $\pi(x)$ is the number of primes $p \le x$. All previous proofs have been by "transcendental" arguments involving some appeal to the theory of functions of a complex variable. Successive proofs have moderated the demands on this theory, or invoked alternative analytical theories (e.g., Fourier transforms), but there remained a nucleus of complex variable theory namely the proposition that the Riemann zeta-function $\zeta(s) = \zeta(\sigma + it)$ has no zeros on the line $\sigma = 1$; and this could hardly be avoided, except by a radically new approach, since the PNT is in a clearly definable sense "equivalent" to this property of $\zeta(s)$. It has long been recognised that an "elementary" proof of the PNT, not depending on analytical ideas remote from the problem itself, would (if indeed possible) constitute a discovery of the first importance for the logical structure of the theory of the distribution of primes. An elementary (though not easy) proof is given, in various forms, in these two papers. Since the papers are in some ways complementary, it seems clearest to review them together.

The starting point is A. Selberg's formula

$$\sum_{p \le x} \log^2 p + \sum_{pq \le x} \log p \log q \sim 2x \log x, \tag{1}$$

where p, q run over primes. The formula might be deduced from the PNT or its equivalent $\vartheta(x) \equiv \sum_{p \le x} \log p \sim x$, but this, on the classical theory of the distribution of primes, would give it a transcendental basis. The significant events to record are: (a) the discovery by Selberg of an elementary proof (indeed with "$\sim 2x \log x$" replaced by "$= 2x \log x + O(x)$," though this refinement is not required for the application); and (b) the recognition that the formula nevertheless embodied facts previously accessible only to the analytical theory of primes. The hope that (1) might therefore be made the basis of an elementary proof of the PNT itself was justified, but the deduction was by no means obvious. The first proof (by Selberg) used, besides (1) (and parts of the elementary theory of primes), the following provisional result (deduced from (1) by Erdös): (2) For any $\lambda > 1$ the number of primes in $(x, \lambda x)$ is at least $Kx/\log x$ for $x > x_0$ ($K = K(\lambda) > 0, x_0 = x_0(\lambda)$). Later it was found possible to argue directly from (1) without the intervention of (2).

Selberg's paper contains his proof of (1), a statement of (2), his deduction of the PNT from (1) and (2), and his final direct deduction of the PNT from (1). Erdös's paper contains a statement of (1), an allusion to Selberg's final proof of the PNT (not published at the time), and an account of other proofs in chronological order; the account includes Erdös's own proof of (2), Selberg's deduction of the PNT from (1) and (2), a sketch of Selberg's simplified proof of (2), and the joint simplified direct deduction of the PNT from (1). It is stated (in both papers) that the method can be adapted to the prime number theorem for arithmetical progressions.

Selberg's proof of (1) is, in accordance with the general spirit of the investigation, expressed in arithmetical form. But in a brief summary it is quicker to treat the formula as the result of equating coefficient-sums in the formal identity $(\zeta'/\zeta)' + (\zeta'/\zeta)^2 = \zeta''/\zeta$, where $\zeta = \zeta(s)$, $\zeta' = \zeta'(s), \cdots$. The left hand side gives substantially the left hand side of (1). As to the right hand side we can write

$$\zeta'' = 2\zeta^3 + b\zeta^2 + c\zeta + \delta,$$

where b, c are certain constants and δ is a Dirichlet's series $\sum d_n n^{-s}$ with coefficient-sum $D(x) \equiv \sum_{n\le x} d_n = O(x^\alpha)$ $(0 < \alpha < 1)$, by elementary theorems on divisor functions. The coefficient-sum of $\zeta''/\zeta = 2\zeta^2 + b\zeta + c + \zeta^{-1}\delta$ can now be estimated with an error $O(x)$, and (1) follows. [This is the general idea; but Selberg, besides avoiding even the formal use of Dirichlet's series, arranges the details rather differently so that the coefficients $d_3(n)$ in ζ^3 do not enter.]

Of the various deductions from (1) the joint simplified proof of the PNT is perhaps the easiest to summarise. In this, (1) is used in the equivalent form

$$\vartheta(x)\log x + \sum_{p\le x}\vartheta(x/p)\log p \sim 2x\log x, \tag{$1'$}$$

which may be rewritten as

$$s(x) + \frac{1}{\log x}\sum_{p\le x}\epsilon_p s(x/p) \to 2, \tag{$1''$}$$

where $s(x) = \vartheta(x)/x$ and $\epsilon_p = (\log p)/p$; and we wish to deduce that $\vartheta(x) \sim x$, i.e., $s(x) \to 1$. We have, by elementary theory: (3) $\sum_{p\le x}\epsilon_p \sim \log x$; so the second term in $(1)''$ is essentially a weighted average of $s(x)$. [The situation now recalls Mercer's theorem; a certain linear combination of a function and its average tends to a (finite) limit, and it is required to deduce that the function itself tends to a limit. The proof (by "compensation") has some features in common with Knopp's proof of Mercer's theorem, but it uses special properties of $s(x)$ that have no counterpart in Mercer's theorem, for example the "Tauberian" condition "$xs(x)$ positive and increasing" and the connection of $s(x)$ with the weights ϵ_p. Suppose that

$$a = \liminf s(x) < \limsup s(x) = A.$$

We have $a > 0$ by elementary theory; and it is an easy deduction from $(1)''$, (3) that: (4) $A + a = 2$. Consider a (large) x for which $s(x)$ is large (i.e., near A). From $(1)''$, (3) we deduce that (to compensate): (α) $s(x/p)$ is small (i.e., near a) for a set X of primes comprising almost all $p \le x$ (where "almost all" means "except for a subset over which $\sum \epsilon_p$ is small compared with the same sum taken over the whole set"). Take a particular (small) p_1 in X. Then similarly: (β) $s(x/p_1p)$ is large for a set Y of primes comprising almost all $p \le x/p_1$. Next, it follows easily from $(1)'$, (4) and the fact that $\vartheta(\cdot)$ is an increasing function that: (γ) points x', x $(x' > x)$ where $s(\cdot)$ is large and small (or vice versa) must be so far apart that $x'/x > (A/a) - \epsilon$ $(= \lambda$, say) for large x. [This is the sense of lemma 5 of

Erdös's paper, but the inequality signs in the conclusion and in most of the proof have become reversed. A list of further misprints (supplied by Erdös) includes: (Lemmas 1, 2) for r read p; (p. 377, line 1) for X read C; (p. 377, line 14 from below) for $\vartheta(x)$ read $\vartheta(x/p)$; (p. 378 (14)) move last square bracket two places to the right; (p. 376, line 9 from below; p. 383, line 3) symbols following $c' > C-$ and $I_i =$ badly printed (but meaning clear).] Now $(\alpha), (\beta), (\gamma)$ involve a contradiction. The proof of this, the most difficult step, is presented in alternative forms. The simpler version is based on a study of the sum $S = \sum \epsilon_p \epsilon_q$ taken over all pairs of primes (p, q) for which p is in X, $\lambda^{-1}p < p_1 q \leq \lambda p$. Writing $S = \sum_p \epsilon_p \sum_q \epsilon_q$, we have $\sum_q \geq (\lambda p/p_1)^{-1}\{\vartheta(\lambda p/p_1) - \vartheta(\lambda^{-1}p/p_1)\} > c > 0$ by the definition of a, A, since $a\lambda > A\lambda^{-1}$; whence $S > (c - \epsilon)\log x$, since, by (α), $\sum_p$ extends over almost all $p \leq x$. On the other hand, writing $S = \sum_q \epsilon_q \sum_p \epsilon_p$, we have

$$\sum_p \leq (\lambda^{-1}p_1 q)^{-1}\vartheta(\lambda p_1 q) < C;$$

whence $S > \epsilon \log x$, since, by (γ), q can never belong to Y (because $x/p_1 q$ is too near to x/p for some p of X), so that, by (β), $\sum_q$ extends over "almost no" primes. This contradiction shows that $a = A$ ($= 1$ since $A + a = 2$). Selberg, in his own final proof, proceeds on somewhat different lines, by "successive approximation." He first derives from (1) an inequality involving (on both sides) the function $R(x) = \vartheta(x) - x$, and then infers from it (and elementary properties of $R(x)$) that, if $|R(x)/x| < \alpha$ ($<$ 8) ($x > x_0$), then $|R(x)/x| < \alpha_1$ ($x > x_1$), where $\alpha_1 = \alpha(1 - k\alpha^2)$ and k is a positive absolute constant; whence by repeated application $|R(x)/x| < \alpha_n$ ($x > x_n$) where $\alpha_n \to 0$ as $n \to \infty$. [The logical arrangement on p. 311 is confusing, since the proof that (3.2) still holds if $R(n)$ changes sign does not seem to be valid until the order of x'/x has been suitably restricted. Also the argument of the text seems to use a sharpened version of (1), though the full form with error $O(x)$ is not required. The claim in the "final remark" that (1) itself suffices would seem to presuppose the use of the alternative form of inequality referred to in footnote 4 on p. 311.]

It may be useful to view the elementary proof against its analytical background, in order to see how the argument, while omitting all reference to analytical facts known to be inseparable from the truth of the PNT, nevertheless avoids the danger of coming into conflict with these facts (and thereby proving too much). Let f be a function of the complex variable $s = \sigma + it$, to be taken as $-\zeta'/\zeta$ in the application but subject at present only to the following restrictions: (i) $f = \sum a_n n^{-s}$ ($\sigma > 1$; a_n real); (ii) f is regular in $\sigma \geq 1$ except possibly for simple poles on $\sigma = 1$; (iii) $g \equiv -f' + f^2 = \sum b_n n^{-s}$ ($\sigma > 1$), where $B(x) \equiv \sum_{n \leq x} b_n \sim 2x \log x$. (When $f = -\zeta'/\zeta$, (iii) is equivalent to Selberg's formula (1).) Then, by (iii), g cannot have a double pole at a point $s \neq 1$ of $\sigma = 1$; whence any possible pole of f at such a point must have residue R satisfying $R + R^2 = 0$, i.e., $R = -1$; and the example

$$f(s) = \zeta(s) - \zeta(s - i\alpha) - \zeta(s + i\alpha)$$

($\alpha > 0$) shows that this is actually possible. Thus our present assumptions are consistent with the existence of poles of f at points $s \neq 1$ on $\sigma = 1$; and this may be taken as a "reason" why it is possible to give an elementary proof of (1) without becoming involved in the question of the existence of zeros of ζ on $\sigma = 1$. Now introduce the further assumption: (iv) $a_n \geq 0$. Then it can be proved, as in Hadamard's classical argument, that a pole of f at $s = 1 + it$ with $R = -1$ implies a pole at $s = 1 + 2it$ with $R = +1$, and this is impossible; so f can have no pole on $\sigma = 1$ except at $s = 1$. Thus the properties (i)-(iv) of $f = -\zeta'/\zeta$ do embody the essential analytical fact on which previous proofs of the PNT have been based. What Selberg and Erdös do is to deduce the PNT directly from the arithmetical counterparts of (i), (iii), (iv), without the explicit intervention of the analytical fact. In principle this opens up the possibility of a new approach, in which the old logical arrangement is reversed and analytical properties of $\zeta(s)$ are deduced from arithmetical properties of the sequence of primes. How far the practical effects of this revolution of ideas will penetrate into the structure of the subject, and how much of the theory will ultimately have to be rewritten, it is too early to say.

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A. E. Ingham

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Selberg, Atle

Collected papers. Vol. I.

With a foreword by K. Chandrasekharan.

Springer-Verlag, *Berlin*, 1989. vi+711 *pp*. \$130.00. *ISBN* 3-540-18389-2

The publication of the collected papers of Atle Selberg is most warmly welcomed by the mathematical community for several reasons. First of all, the author is a living classic who has profoundly influenced mathematics, especially analytic number theory in a broad sense, for about fifty years. Secondly, his papers up to 1947, which appeared mostly in Norwegian series or journals of limited distribution and partly even during World War II, are now at last easily accessible. And thirdly, a lot of highly interesting mathematics comes into daylight via the two volumes of Selberg's collected papers; in fact, Volume II contains entirely unpublished material! Even the present volume contains a couple of previously unpublished papers, in addition to all the author's publications covering the period 1936–1988. The latter works, constituting the bulk of Volume I, contain outstanding contributions such as the Rankin-Selberg method, the "mollifier" device in the theory of Riemann's zeta function with its deep applications to zeros on or near the critical line and with Selberg's sieve as a by-product; further, the elementary proof of the prime number theorem, Selberg's trace formula, Selberg's zeta function,...just to mention a few highlights.

Papers 39 and 41, entitled "Harmonic analysis" and "Reflections around the Ramanujan centenary", appear here in print for the first time. The former contains the most essential part (the case of a discrete group with a noncompact fundamental domain of a finite measure) of Selberg's Göttingen lectures in 1953 on his trace formula. Though unpublished, these lectures have been more or less known to experts, and it is definitely a good idea to make them now more widely accessible, with an introduction and comments by the author. The latter of the two papers mentioned above is of a more informal and philosophical nature. It contains the author's extemporaneous talk at the conclusion of the Ramanujan Centenary Conference in January 1988 at the Tata Institute in Bombay, together with an appendix concerning Rademacher's identity for the partition function. Reflections about Ramanujan's decisive role in his mathematical development and about the nature of mathematics in general provide an exciting glimpse into Selberg's thinking. As early as a schoolboy, he became deeply impressed by Ramanujan's *Collected papers*, feeling in it and in Ramanujan's personality "the air of mystery", which gave him the first impetus to his own mathematical explorations. As another impulse of lasting influence, he mentions Hecke's lecture at the International Mathematical Conference in Oslo in 1936. From the perspective of such underlying "Jugendträume", one is in a position to appreciate as if from the inside the momumental architecture of the author's creative work.

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Matti Jutila

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Selberg, Atle

Collected papers. Vol. II.

With a foreword by K. Chandrasekharan.

Springer-Verlag, *Berlin*, 1991. viii+253 *pp*. \$103.00. *ISBN* 3-540-50626-8

While Volume I [*Collected papers. Vol. I*, Springer, Berlin, 1989; MR1117906 (92h:01083)] consisted mainly (though not entirely) of work which had been previously published in journals or conference proceedings, the material in the present Volume II is more or less "new" in a bibliographic sense. However, the results or ideas as such often go back to much earlier times, and many of them have been mentioned by Selberg in his lectures or private communications. In the afterword, he says that the publication of his collected works gave him "a good occasion for writing up various things that over the years had remained unpublished", and that "this second volume contains what I found time to finish, the major part being the lectures on sieves, which I take some satisfaction in having completed".

Beside these sieve lectures, there are three other papers: "Linear operators and automorphic forms", "Remarks on the distribution of poles of Eisenstein series" [in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, 251–278, Weizmann, Jerusalem, 1990; MR1159119 (93c:11035)] and "Old and new conjectures about a class of Dirichlet series" [in *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, 367–385, Univ. Salerno, Salerno, 1992; MR1220477 (94f:11085)], which have recently appeared in various conference volumes. For these papers, we refer to their individual reviews.

As to the lectures on sieve theory, the author describes his goals as follows: "Beside the general theory, I give a full account of my theory of the Buchstab-Rosser sieve—the results are of course in agreement with those independently found by Iwaniec—as well as more details on the Λ^2 sieve and the $\Lambda^2\Lambda^-$ sieve." The basic sieve problem is formulated in terms of weighted sets and variable (rather than constant) sifting density. In addition to the general theory, there are interesting historical remarks and applications to special problems such as the twin prime and Goldbach conjectures and the Brun-Titchmarsh theorem.

The present volume covers the core of Selberg's lasting mathematical interest: automorphic functions, Dirichlet series and sieves. No doubt it will stimulate (and has already stimulated) further work in these fields.

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