BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 1, January 2009, Pages 35–54
S 0273-0979(08)01229-9
Article electronically published on September 22, 2008

CONFORMAL INVARIANCE AND $2D$ STATISTICAL PHYSICS

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Abstract. A number of two-dimensional models in statistical physics are conjectured to have scaling limits at criticality that are in some sense conformally invariant. In the last ten years, the rigorous understanding of such limits has increased significantly. I give an introduction to the models and one of the major new mathematical structures, the Schramm-Loewner Evolution (SLE).

1. Critical phenomena

Critical phenomena in statistical physics refers to the study of systems at or near the point at which a phase transition occurs. There are many models of such phenomena. We will discuss some discrete equilibrium models that are defined on a lattice. These are measures on paths or configurations where configurations are weighted by their energy $\mathcal{E}$ with a preference for paths of smaller energy. These measures depend on at least one parameter. A standard parameter in physics is $\beta = c/T$, where c is a fixed constant, T stands for temperature, and the measure given to a configuration is $e^{-\beta\mathcal{E}}$. Phase transitions occur at critical values of the temperature corresponding to, e.g., the transition from a gaseous to a liquid state. We use β for the parameter although for understanding the literature it is useful to know that large values of β correspond to "low temperature" and small values of β correspond to "high temperature". Small β (high temperature) systems have weaker correlations than large β (low temperature) systems. In a number of models, there is a critical β_c such that qualitatively the system has three regimes $\beta < \beta_c$ (high temperature), $\beta = \beta_c$ (critical), and $\beta > \beta_c$ (low temperature).

A standard procedure is to define a model on a finite subset of a lattice, and then let the lattice size grow. Equivalently, one can consider a bounded region and consider finer and finer lattices inside the region. In either case, one would like to know the scaling or continuum limit of the system. For many models, it can be difficult to even describe what kind of object one has in the limit, and then it can be much more difficult to prove such a limit exists.

The behavior of the models described iin this paper varies significantly in different dimensions. In two dimensions, Belavin, Polyakov, and Zamolodchikov predicted[1] [3, 4] that many systems at criticality (at β_c) had scaling limits that were in

Received by the editors June 20, 2008.
2000 *Mathematics Subject Classification.* Primary 82B27; Secondary 30C35, 60J65, 82B27.
This research was supported by National Science Foundation grant DMS-0734151.

[1] I use the word "predicted" to mean that the result was mathematically nonrigorous but had significant theoretical argument behind it. There is much nontrivial, deep mathematics in the conformal field theory arguments and other theories in mathematical physics. I use the word "predict" rather than "conjecture" to acknowledge this.

some sense conformally invariant. This assumption and the nonrigorous techniques of conformal field theory allowed for exact calculation of a number of "critical exponents" and other quantities in the limit. For mathematicians, many of the arguments were unsatisfactory, not only because there was incomplete proof, but also because in many cases there was no precise statement of what kind of limit was taken or what the limiting object was. However, the predictions made from these arguments were very consistent with numerical simulations, so it was clear that there was something essentially correct about the arguments.

There are a number of different approaches to studying two-dimensional critical phenomena. This paper will focus on the progress in understanding the geometric and fractal properties of the scaling limit. The big step was the introduction of the Schramm-Loewner evolution (or stochastic Loewner evolution as Schramm called it) which is a measure on continuous curves. The definition combines work in classical function theory by Loewner with the fundamental quantity in stochastic analysis, Brownian motion.

2. Lattice models

2.1. **Self-avoiding walk.** A self-avoiding walk (SAW) in the lattice $\mathbb{Z}^2$ is a nearest neighbor path that has no self-intersections. SAWs arose as a model of polymer chains (in a dilute solution). Roughly speaking, a single polymer chain consists of a number of monomers that take a random shape with the only constraint being that the polymer cannot cross itself. We put a measure on SAWs for which all walks of the same length have the same measure. More precisely, we assign measure $e^{-\beta n}$ to each walk

$$\omega = [\omega_0, \ldots, \omega_n],$$

with $|\omega_j - \omega_{j-1}| = 1$ for each j and $\omega_j \neq \omega_k, j < k$. Trying to understand the SAW problem leads to the following (easily stated but still open) problems:

- How many SAWs are there of length n with $\omega_0 = 0$?
- If one chooses a SAW at random from the set of SAWs of length n, what is the typical end-to-end (Euclidean) distance of the chain?

It is conjectured that there exist c, β_c, γ such that the number of walks of length n, C_n satisfies

$$C_n \sim c\, e^{\beta_c n}\, n^{\gamma - 1}.$$

A simple subadditivity argument shows the existence of a β_c such that $\log C_n \sim \beta_c n$, but the more precise asymptotics are still open questions. The number γ is one of the "critical exponents" for the problem. Another critical exponent, usually denoted ν, states that the average end-to-end distance grows like n^ν.

Let us consider SAWs with boundary conditions. In Figure 1, we have an $N \times N$ square in $\mathbb{Z}^2$ and let z, w be boundary points on opposites sides. Consider the set of SAWs starting at z ending at w and otherwise staying in the box. We give each such walk ω measure $e^{-\beta|\omega|}$, where $|\cdot|$ denotes the length (number of edges) in the walk ω. The total mass of this measure

$$Z_N = Z_{N,\beta} = \sum_{\omega : z \to w} e^{-\beta|\omega|}$$

is often called the *partition function*. For any N, β, we get a probability measure on paths by dividing by Z_N. For large N, the behavior of this probability measure varies depending on β:

- If $\beta < \beta_c$, then Z_N grows exponentially in N. The penalty for having many bonds is not high, and a typical path tends to fill up the square.
- If $\beta > \beta_c$, then Z_N decays exponentially in N. The penalty for having many bonds is high, and a typical path goes from z to w without visiting many more sites than necessary.
- If $\beta = \beta_c$, then Z_N neither grows nor decays exponentially. It is expected that it decays like a power of N. The typical path is a typical SAW path of $N^{1/\nu}$ steps and roughly looks $(1/\nu)$-dimensional.

Let us focus on the critical value $\beta = \beta_c$. If we scale space and time, then we might hope to get a probability measure on continuous paths connecting two boundary points of a square. This measure would be supported on curves whose fractal dimension is $1/\nu$.

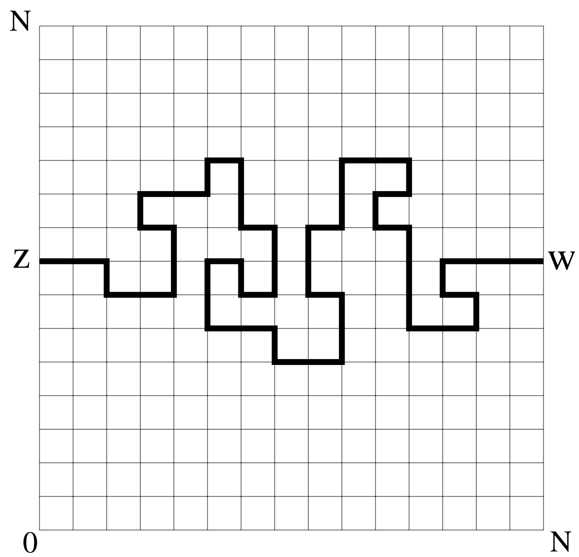

FIGURE 1. Self-avoiding walk in a domain

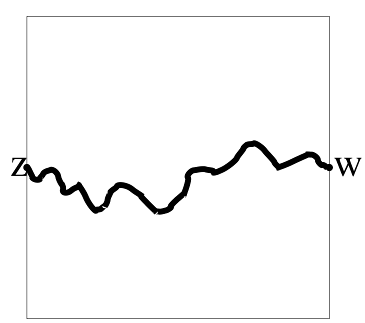

FIGURE 2. Scaling limit of SAW

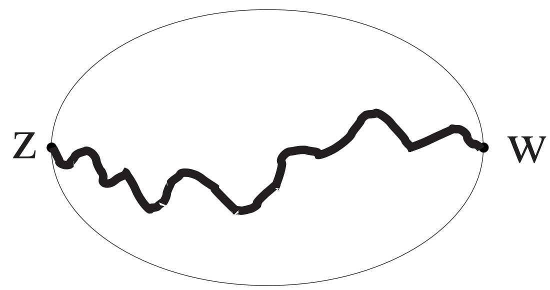

FIGURE 3. Scaling limit of SAW in a different domain

The existence of this limit for SAW has not been proved. However, let us suppose that such a limit exists. Suppose we took a different domain with two boundary points and considered a similar limit measure at $\beta = \beta_c$. This gives a measure on curves in the new domain. If the new domain is simply connected, we can also get a probability measure on curves by starting with the measure on the square and mapping these curves to the new domain. (There are two issues we are not dealing with. One is the local lattice effect at the boundary if the boundary does not match up nicely with the lattice as in the case of the square above. The other is the parametrization of the curves. We will consider two curves the same if one is an increasing reparametrization of the other). Conformal invariance would imply that one can obtain one measure from the other by means of a conformal transformation.

2.2. **Simple random walk.** There is a variation of the last model that is much better understood by probabilists. Suppose we do not put on a self-avoidance constraint and consider all nearest-neighbor random walks. Then the number of walks of length n is 4^n and hence $e^{\beta_c} = 4$. The typical end-to-end distance of a simple random walk is $O(n^{1/2})$, i.e., $\nu = 1/2$. For a simple random walk, one can show that the partition function for the square in Figure 4 with $\beta = \beta_c$ is comparable to N^{-2}. When we scale the paths as above, we get a well-known limit, known as *Brownian motion.* (More precisely, we get a Brownian "excursion" from z to w—this is a modification of Brownian motion for which the path starting at z goes immediately into the domain and then exits the domain at w.)

It is known that the scaling limit of a Brownian excursion is conformally invariant. The result for Brownian motion goes back to Paul Lévy [32]. In fact, it is implicit in earlier work—the basic fact is that harmonic functions in two dimensions are invariant under conformal transformations.

2.3. **Loop-erased random walk.** The loop-erased random walk (LERW) is the measure obtained from a simple random walk by erasing loops. In other words, we start with the simple random walk measure and for each walk ω we obtain a walk without self-intersections by erasing the loops. (The walk that one obtains from loop-erasure depends on the order in which the points are erased—in this case, we specify that the loops are erased chronologically.) The LERW arises in a number of situations; e.g., as the distribution of a typical geodesic in a spanning tree of a graph where the tree is chosen from the uniform distribution of all spanning trees.

Since the partition function (the sum of the weights of all the paths) is the same as for simple random walk, it has the same β_c and partition function. However,

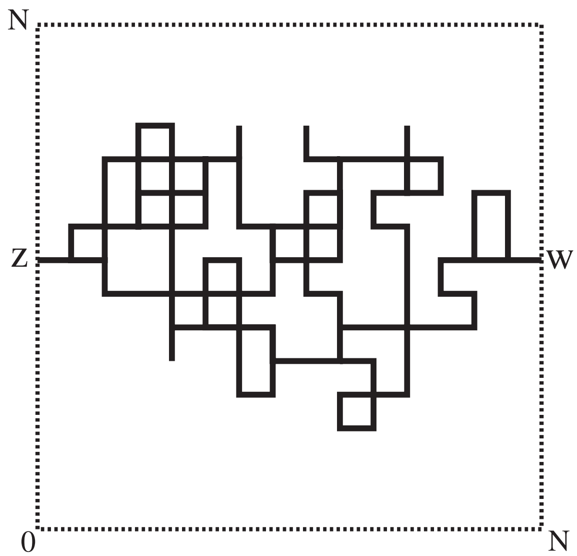

FIGURE 4. Simple random walk in D

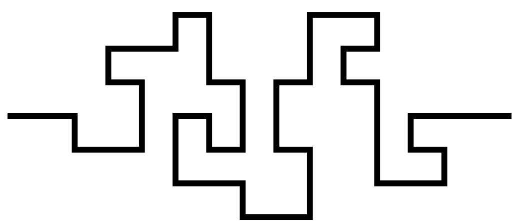

FIGURE 5. The walk obtained from erasing loops chronologically from the simple walk above

the number of bonds in a typical LERW is smaller than that for the simple random walk. It was predicted by physicists (and a version proved by Rick Kenyon [19]) that the typical LERW has on the order of $N^{5/4}$ bonds. If we take a scaling limit, then we would expect that the paths in the scaling limit would have fractal dimension 5/4.

We also would conjecture that the LERW gives a conformally invariant limit. For example, one could use the fact that a simple random walk has a conformally invariant limit and note that the loop-erasing procedure, which only looks at the order of the points, should also be conformally invariant. There are other relations between the LERW and a simple random walk that would lead us to conjecture this.

It is not so easy to give a formula for the weight of any particular path ω under this measure. For each self-avoiding path, the LERW measure of the path is the measure of all the simple random walks whose loop-erasure is ω. It turns out that one can write this weight as

$$4^{-|\omega|}\, e^{\Lambda(\omega)},$$

where $\Lambda(\omega)$ is a measure of the number of loops in the domain that intersect ω. The scaling limit of this loop measure arises in studying the scaling limit of a number of models.

2.4. **Percolation.** Percolation can be considered a model of permeability of a material. We will describe a lattice model for percolation on the triangular lattice. Suppose that every point in the triangular lattice in the upper half-plane is colored black or white independently with white having probability p. A typical realization with $p = 1/2$ is illustrated in Figure 6 (if one ignores the bottom row). We think of a white site as being "open" through which liquid can flow. The general question is whether or not there is an infinite collection of open sites that are connected. The value $p = 1/2$ is "critical" for the triangular lattice in that for $p > 1/2$, there will be an infinite connected cluster of white sites while for $p < 1/2$, this will not be true. We will consider critical percolation, $p = 1/2$.

We now put a boundary condition on the bottom row as illustrated—all black on one side of the origin and all white on the other side. Once all the other colors in the upper half-plane have been chosen, there is a unique curve starting at the bottom row that has all white vertices on one side and all black vertices on the other side. This is called the *percolation exploration process*. Similarly we could start with a domain D and two boundary points z, w, give a boundary condition of black on one of the arcs and white on the other arc, put a fine triangular lattice inside D, color vertices black or white independently with probability 1/2 for each, and then consider the path connecting z and w. In the limit, one might hope for a continuous interface.

There is another conformal invariant for percolation first predicted by John Cardy [7, 8]. Suppose D is a simply connected domain and the boundary is divided into four arcs, A_1, A_2, A_3, A_4, in counterclockwise order. Let $P_D(A_1, A_3)$ be the limit as the lattice spacing goes to zero of the probability that in a percolation cluster as above there is a connected cluster of white vertices connecting A_1 to A_3.

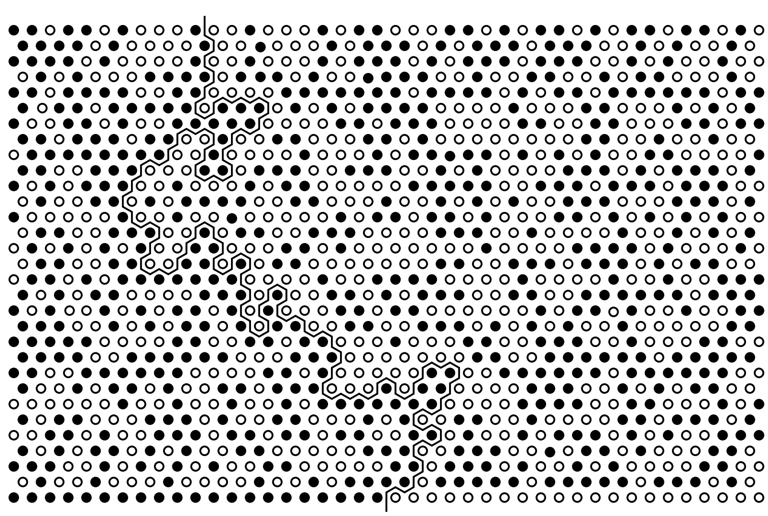

Figure 6. The percolation exploration process.

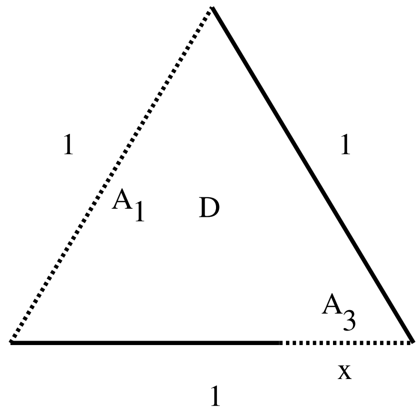

FIGURE 7. Cardy's formula: $P_D(A_1, A_3) = x$.

This should be a conformal invariant. It turns out that the nicest domain to give the formula is an equilateral triangle; see Figure 7.

2.5. **Ising model.** The Ising model is a simple model of a ferromagnet. Consider the triangular lattice as in Figure 6. Again we color the vertices black or white although we now think of the colors as spins—black is a spin up and white is a spin down. If x is a vertex, we let $\sigma(x) = 1$ if x is colored black and $\sigma(x) = -1$ if x is colored white. The measure on configurations is such that neighboring spins like to be aligned with each other.

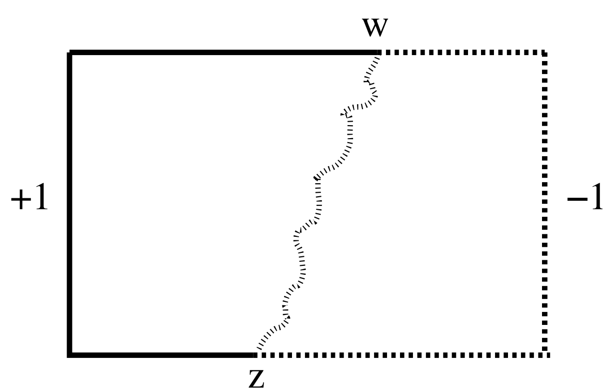

FIGURE 8. Ising interface.

It is easiest to define the measure for a finite collection of spins. Suppose D is a bounded domain in $\mathbb{C}$ with two marked boundary points z, w which give us two boundary arcs. We consider a fine lattice in D and fix boundary conditions $+1$ and -1, respectively, on the two boundary arcs. Each configuration of spins is given energy

$$\mathcal{E} = -\sum_{x \sim y} \sigma(x)\, \sigma(y),$$

where $x \sim y$ means that x, y are nearest neighbors. We then give measure $e^{-\beta \mathcal{E}}$ to a configuration of spins. If β is small, then the correlations are localized and spins separated by a large distance are almost independent. If β is large, there is long-range correlation. There is a critical β_c that separates these two phases. At this critical value, the interface is a random fractal.

3. Assumptions on the limit

In trying to find possible scaling limits for models as above, Schramm started by giving assumptions that a scaling limit of the curves arising in critical systems should satisfy. Suppose there exists a family of probability measures $\mu(D; z, w)$, indexed by a collection of domains D and distinct boundary points $z, w \in \partial D$, supported on curves connecting z and w in D. We assume that the family satisfies conformal invariance.

- **Conformal invariance.** If $f : D \to f(D)$ is a conformal transformation, then
$$f \circ \mu(D; z, w) = \mu(f(D); f(z), f(w)).$$

To be more precise, if γ is a curve in D from z to w, then $f \circ \gamma(t) = f(\gamma(t))$ gives a curve in $f(D)$ from $f(z)$ to $f(w)$. We adopt the convention that two curves are equivalent if one is an (increasing) reparametrization of the other. For this reason, we do not worry about the parametrization of $f \circ \gamma$. If we have a measure on curves γ in D, then f induces a measure on curves $\tilde{\gamma}$ in $f(D)$ by the map $\gamma \mapsto f \circ \gamma$.

We now consider all the examples in the previous section, except the simple random walk (for which we know the scaling limit, Brownian motion). For all the other systems, the discrete models satisfy a property called the *domain Markov property* which we would expect to be satisfied by the scaling limit. We will state this property under the assumption that the curves γ are simple (non-self-intersecting). Suppose we are interested in the distribution of the curve γ, and we observe an initial part of the curve $\gamma(0, t]$. Let D_t denote the slit domain $D \setminus \gamma(0, t]$. (More generally, if γ can have self-intersections, we let D_t denote the connected component of $D \setminus \gamma(0, t]$ that contains w on the boundary. We require our limit to satisfy the "noncrossing" condition that $\gamma(t) \in \partial D_t$. This is not satisfied for the Brownian excursion.)

- **Domain Markov property.** Given $\gamma[0, t]$ the distribution of the remainder of the path is given by $\mu(D_t; \gamma(t), w)$.

Schramm showed that if we restrict to simply connected domains D, then there is only a one-parameter family of probability measures that satisfy both conformal invariance and the domain Markov property. We will explain why in the next two sections. The Riemann mapping theorem tells us that all simply connected domains in $\mathbb{C}$ with nontrivial boundary are conformally equivalent. In particular, if D is a simply connected domain with distinct boundary points z, w, then there exists a conformal transformation

$$F : \mathbb{H} = \{x + iy : y > 0\} \to D, \quad F(0) = z, \quad F(\infty) = w.$$

The map is not unique, but if $\tilde{F}$ is another such transformation, then $\tilde{F}(z) = F(rz)$ for some $r > 0$. If we can define the measure $\mu(\mathbb{H}; 0, \infty)$, then the measure $\mu(D; z, w)$ for all simply connected D and distinct $z, w \in \partial D$ will be determined.

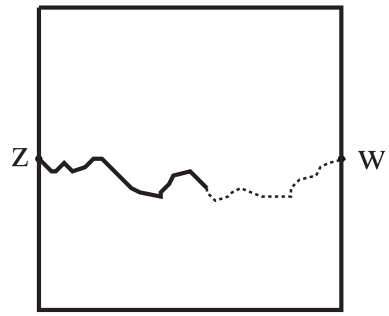

Figure 9. Domain Markov property

4. Loewner differential equation

One of the biggest problems of classical function theory in the twentieth century was the Bieberbach conjecture. The Riemann mapping theorem showed that the study of simply connected domains reduces to the study of univalent, i.e., one-to-one and analytic, functions on the unit disk $\mathbb{D}$. By translation, dilation, and rotation, it suffices to consider the set $\mathcal{S}$ of such functions with $f(0) = 0, f'(0) = 1$. Each such function can be written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n.$$

Ludwig Bieberbach proved that $|a_2| \leq 2$ and conjectured that $|a_n| \leq n$ for all n. (There exists a particular f that obtains the maximal values.) Compactness arguments show that it suffices to prove the inequality for slit domains of the form $\mathbb{C} \setminus \gamma[t, \infty)$, where γ is a simple curve with $\gamma(z) \to \infty$, $\gamma(-\infty) = 0$. Charles Loewner[2] [33] derived a differential equation that described the dynamics of the coefficients $a_n(t)$ in t. He used this to prove $|a_3| \leq 3$, and the general technique became an important tool in the study of conformal maps. In fact, the proof of the Bieberbach conjecture by Louis de Branges [5] uses such chains.

There are a number of versions of the Loewner differential equation. Schramm found it most convenient to consider a version in the upper half-plane $\mathbb{H}$. Suppose $\gamma : (0, \infty) \longrightarrow \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$. For each t, let H_t denote the slit domain $\mathbb{H} \setminus \gamma(0, t]$. Using the Riemann mapping theorem, one can show that there is a unique conformal transformation g_t of H_t onto $\mathbb{H}$ satisfying $g_t(z) = z + o(1)$ as $z \to \infty$. This can be expanded at infinity as

$$g(z) = z + \frac{a(t)}{z} + O(|z|^{-2}),$$

where $a(t) \geq 0$ depends on the curve $\gamma(0, t]$. The quantity $a(t)$ is a "half-plane capacity". It can be shown that $t \mapsto a(t)$ is a continuous, strictly increasing function. By making a slightly stronger assumption on the curve γ, we can also assume that $a(t) \to \infty$. Since a is strictly increasing, we can reparametrize the curve γ so that

[2]He spelled his name Karel (in Czech) or Karl (in German) Löwner in Europe but adopted the spelling Charles Loewner when he moved to the United States.

$a(t)$ grows linearly, say $a(t) = 2t$. If we do this, then the (chordal or half-plane) Loewner differential equation states that the function $t \mapsto g_t(z)$ satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{4.1}$$

where $U_t = g_t(\gamma(t))$. (Although $\gamma(t) \in \partial H_t$, one can show that there is a unique continuous extension of g_t to the boundary at $\gamma(t)$.) Moreover, the function $t \mapsto U_t$ is a continuous function from $[0, \infty)$ to $\mathbb{R}$. For $z \in \mathbb{H}$, the solution exists up to time $T_z \in (0, \infty]$. In fact, for a simple curve γ, $T_z = \infty$ if $z \notin \gamma(0, \infty)$ and $T_{\gamma(t)} = t$. To understand this equation, let us consider $t = 0$,

$$\partial_t g_t(z)\Big|_{t=0} = \frac{2}{z}.$$

The function $z \mapsto 1/z$ is (a multiple of) the complex form of the Poisson kernel in the upper half-plane. The Loewner equation states that if one parametrizes the curve by capacity, then the change in the conformal map g_t is determined by the Poisson kernel.

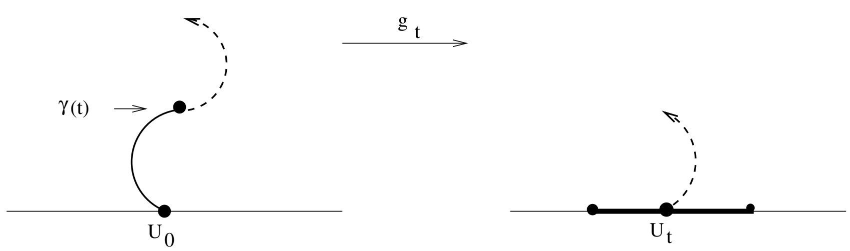

FIGURE 10. The conformal map g_t

If we start with a continuous function $t \mapsto U_t$ on the real line, then for each $z \in \mathbb{H}$, we can find a solution to the equation (4.1) that exists up to a time $T_z \in (0, \infty]$. In fact, for fixed t, the function g_t is the unique conformal transformation of $H_t := \{z : T_z > t\}$ onto $\mathbb{H}$ that satisfies $g_t(z) - z = o(1)$ as $z \to \infty$. However, it is not always the case that the domain H_t will be a simple slit domain as above.

5. Schramm-Loewner evolution

Suppose we have a random simple curve $\gamma(t)$ in $\mathbb{H}$ that arises from a family of curves satisfying conformal invariance and the domain Markov property. The random curves $\gamma(t)$ generate random one-dimensional continuous functions U_t by (4.1). Conformal invariance and the domain Markov property translate into assumptions on U_t:

- For each $s < t$, the random variable $U_t - U_s$ is independent of $U_{s'}, 0 \leq s' \leq s$ and has the same distribution as U_{t-s}.

Since the measure $\mu(\mathbb{H}; 0, \infty)$ should be invariant under dilations $z \mapsto rz$, the function $r\gamma(t)$ should have the same distribution (modulo reparametrization) as $\gamma(t)$. Using properties of capacity, we can see that if we set $\tilde{\gamma}(t) = r\gamma(t/r^2)$, then $\tilde{\gamma}(t)$ is parametrized by capacity. Using this and doing the appropriate change of variables, we also get

- The distribution of rU_{t/r^2} should be the same as the distribution of U_t.

It is well known to probabilists that there is only a one-parameter family of processes U_t that satisfy these conditions,

$$U_t = \sqrt{\kappa}\, W_t,$$

where W_t is a standard Brownian motion. One can choose any $\kappa > 0$. This leads to Schramm's definition.

Definition 5.1. *(Chordal) SLE_κ (from 0 to ∞ in $\mathbb{H}$)* is the random collection of conformal maps g_t obtained by solving (4.1) with $U_t = \sqrt{\kappa}\, W_t$.

Chordal SLE_κ in other simply connected domains is defined by conformal transformation. The invariance of SLE_κ in $\mathbb{H}$ under dilations shows that this definition is independent of the choice of map. It is not obvious, but was proved [35] that there is a random function $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that for each t, g_t is the conformal transformation of the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$. This curve is called the SLE_κ curve or SLE_κ trace. If D is a simply connected domain and z, w are distinct boundary points, then SLE_κ in D from z to w is obtained by conformal transformation.

It turns out that the qualitative behavior of the curve varies greatly as the parameter κ varies.

- If $0 < \kappa \leq 4$, the curve γ is simple and $\gamma(0,\infty) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, the curve γ has self-intersections but is "noncrossing" and does not fill up the plane. Noncrossing implies that for all t, $\gamma(t)$ is on the boundary of the unbounded connected component of $\mathbb{H} \setminus \gamma(0,t]$. γ also intersects $\mathbb{R} \setminus \{0\}$.
- If $\kappa \geq 8$, the curve is noncrossing and plane-filling! For each $z \in \overline{\mathbb{H}}$ there is a t such that $\gamma(t) = z$.

Moreover [2, 35], the Hausdorff dimension of the paths is given by a simple formula

$$d_\kappa := \dim(\gamma[0,t]) = 1 + \frac{\kappa}{8}, \qquad t > 0, \quad \kappa \leq 8. \tag{5.1}$$

For $\kappa \geq 8$, the dimension is two since the curve is plane-filling. The relation $\kappa \leftrightarrow d_\kappa$ is a bijection between $[0,8]$ and $[1,2]$. ($\kappa = 0$ corresponds to a straight line, i.e., the curve $\gamma(t) = cti$. It is the solution of the Loewner equation with $U_t \equiv 0$.)

Fix $z \in \overline{\mathbb{H}} \setminus \{0\}$ and let $\tilde{Z}_t = \tilde{Z}_t(z) = g_t(z) - U_t$. Then the Loewner equation (4.1) becomes the stochastic differential equation (SDE)

$$d\tilde{Z}_t = \frac{2}{\tilde{Z}_t}\, dt + \sqrt{\kappa}\, dB_t,$$

where $B_t = -W_t$ is a standard Brownian motion. If we do the time change $Z_t = \tilde{Z}_{t/\kappa}$, this equation becomes

$$dZ_t = \frac{a}{Z_t}\, dt + dW_t, \tag{5.2}$$

where W_t is a standard Brownian motion and $a = 2/\kappa$. This SDE is called the Bessel equation. It is well known that for real Z_t, if $a \geq 1/2$ ($\kappa \leq 4$), the solutions to this equation never reach the origin, i.e., the push away from the origin of a/Z_t is big enough to withstand the randomness in the Brownian motion W_t. On the other hand, if $a < 1/2$ ($\kappa > 4$), the solutions eventually hit the origin. Geometrically, this corresponds to the fact that for $\kappa \leq 4$, the point 1 stays on the boundary of H_t which is the same as saying that the curve $\gamma(0,\infty)$ never hits $(1,\infty)$.

The basic tool for studying SLE is stochastic calculus and much can be learned using relatively simple ideas. The fundamental theorem of stochastic calculus is *Itô's formula*. Roughly, it states that if X_t is a process satisfying

$$dX_t = A_t\, dt + R_t\, dW_t,$$

where W_t is a standard Brownian motion and $f(t,x)$ is a function, then

$$df(t,X_t) = \left[\partial_t f(t,X_t) + \frac{R_t^2}{2}\,\partial_{xx} f(t,X_t)\right] dt + A_t\,\partial_x f(t,X_t)\, dW_t.$$

We can think of this as the usual chain rule with the added rule $dW_t^2 = t$.

We give an example of how stochastic calculus is used to compute a probability for SLE. Let $E = E_z$ be the event that z lies on the "left side" of the SLE path $\gamma(0,\infty)$. By scaling, $\mathbf{P}(E_z)$ depends only on $\arg(z)$ and hence we can write $\mathbf{P}(E) = f(\arg z)$ for some function f with $f(0) = 0, f(\pi) = 1$. If

$$J_t = \arg(Z_t),$$

then $f(J_t)$ denotes the conditional probability of the event E given $\gamma(0,t]$. In probabilistic notation

$$f(J_t) = \mathbf{P}[E_z \mid \mathcal{F}_t],$$

where $\mathcal{F}_t$ denotes the "information" contained in $\gamma(0,t]$. The process $M_t = f(J_t)$ is an example of a *martingale*, a process which satisfies $\mathbf{E}[M_t \mid \mathcal{F}_s] = M_s$ for $s < t$. Itô's formula and (5.2) give

$$dJ_t = (1-2a)\,\cos J_t\,\sin J_t\, dt - \sin J_t\, dB_t.$$

and another application of Itô's formula gives

$$df(J_t) = \left[(1-2a)\,\cos J_t\,\sin J_t\, f'(J_t) + \frac{1}{2}\sin^2 J_t f''(J_t)\right] dt + [\cdots]\, dB_t.$$

If we have a process such as $f(J_t)$ written in terms of a stochastic differential equation as above and we know that $f(J_t)$ is a martingale, then it must follow that the dt term is zero, i.e., that $f(\theta)$ satisfies

$$2(1-2a)\, f'(\theta)\,\cot\theta + f''(\theta) = 0,$$

which yields

$$f(\theta) = \int_0^\theta \frac{c\, dr}{\sin^{2-4a} r},$$

where c is chosen so that $f(\pi) = 1$. Note that the integral is finite only if $2-4a < 1$, i.e., if $\kappa < 8$. In the plane-filling regime $\kappa \geq 8$, the point z is hit and hence is not on the right or left side.

We see that facts about the path can often be reduced to questions about stochastic differential equations. Let us give another example, determining the fractal dimension of the curve γ for $\kappa < 8$. Roughly speaking, we say that γ has fractal dimension d if for each $0 \ll t < \infty$, $\gamma[s,t]$, the number of balls of radius ϵ needed to cover $\gamma[s,t]$ grows like ϵ^{-d} as $\epsilon \to 0$. We can cover a ball of radius 1 with $O(\epsilon^{-2})$ balls of radius ϵ. The *expected number* of these balls that intersect the path $\gamma[s,t]$ should be of order ϵ^{-d}, and therefore the probability that a particular ball intersects $\gamma[0,1]$ should decay like ϵ^{2-d}. This leads us to ask the following question: Is there a function $G(z)$ and a d such that for each $z \in \mathbb{H}$, the probability that $\gamma[0,\infty)$ enters $\mathcal{B} = \mathcal{B}(z,\epsilon)$, the disk of radius ϵ about z, is about $G(z)\,\epsilon^{2-d}$? Let $E = E_{z,\epsilon}$ be this probability and consider the conditional probability of E given $\gamma[0,t]$, where t is sufficiently small so that $\gamma[0,t] \cap \mathcal{B} = \emptyset$. The conditional probability is the probability that an SLE starting at W_t visits $g_t(\mathcal{B})$, which is approximately the disk of radius $\epsilon\,|g_t'(z)|$ about $g_t(z)$. This should be about $\epsilon^{2-d}\,|g_t'(z)|^{2-d}\,G(g_t(z)-W_t)$. With this in mind, we try to find d, G such that

$$M_t = |g_t'(z)|^{2-d}\,G(g_t(z)-W_t)$$

is a martingale. Using Itô's formula again, one finds that

$$d = 1+\frac{\kappa}{8}, \qquad G(x+iy) = y^{d-2}\,(x^2+1)^{\frac{1}{2}-\frac{4}{\kappa}}.$$

(A technical issue arises in that M_t is not strictly a martingale but is what is known as a *local martingale*; we will not worry about this here.) While this calculation is straightforward, it takes a significant amount of work [2] to prove that the Hausdorff dimension of the paths is $1+\frac{\kappa}{8}$.

6. Central charge and other parameters

Families of curves on simply connected domains that satisfy conformal invariance and the domain Markov property must be SLE curves. However, there is a one-parameter family of such curves. For a particular model, one must determine what is the value of κ that corresponds to that model. Sometimes, this can be determined by computing some quantity for that model or demonstrating some property that would need to be satisfied in the limit. For example, if we knew the Hausdorff dimension of the curves in the limit, then if this dimension is less than two, we would know κ by (5.1). In practice, the dimension is generally one of the properties that we are trying to determine about the limiting curves, so we could not use this to find κ.

Another quantity is the *(boundary) scaling exponent* which is sometimes called the *conformal weight* or a *scaling dimension*. This exponent can be defined directly in terms of SLE but it corresponds to an exponent we have seen before. Recall the self-avoiding walk and the loop-erased random walk. The partition function at $\beta=\beta_c$ is conjectured to satisfy a power law, $Z_N \sim N^{-2b}$ for an exponent b. For SLE_κ, this exponent corresponds to the the boundary scaling exponent that can be computed

$$b = \frac{6-\kappa}{2\kappa}.$$

As mentioned before, properties of a simple random walk determine the behavior of its (and hence also the LERW) partition function, which lead us to identify $b=1$ or $\kappa=2$ for a loop-erased walk. In the case of critical percolation, the partition function is actually equal to 1 because the weights of configurations are determined by a probability measure. This would lead one to guess $b=0$ or $\kappa=6$. For other examples, such as SAW, the scaling exponent is something one would like to compute for the model.

In conformal field theory there is a constant $\mathbf{c}$, called the *central charge*, which is used to describe the theory. (The term central comes from its relationship to central extensions of Lie algebra which we will not discuss here.) Bertrand Duplantier was the first to conjecture the following relationship between $\mathbf{c}$ and κ:

$$\mathbf{c} = \frac{(3\kappa-8)\,(6-\kappa)}{2\kappa}, \qquad \kappa = \frac{(13-\mathbf{c}) \pm \sqrt{(13-\mathbf{c})^2-144}}{3}.$$

Note that the relationship $\kappa \leftrightarrow \mathbf{c}$ is two-to-one except at the double root $\mathbf{c} = 1, \kappa = 4$. Also, for $\kappa > 0$, $c \leq 1$. We define $\kappa' = 16/\kappa$ to be the "dual value" of the SLE parameter; κ and κ' have the same central charge $\mathbf{c}$. If $\kappa < 4$, then the SLE paths are simple, but the curves for the dual value $\kappa' > 4$ are not simple. There is a duality relation that states that if $\kappa \leq 4$ and γ is an $SLE_{\kappa'}$ curve, then for each t the "outer boundary" of the curve (which can be defined as $\mathbb{H} \cap \partial H_t$ where H_t is as above) looks like an SLE_κ curve. For certain values of κ this follows from properties of particular models. Versions of the duality relation for all κ were established recently by Julien Dubédat [13] and Dapeng Zhan [45] independently.

One can give an interpretation of the central charge in terms of how the measure on configurations changes when a domain is perturbed. Suppose $\kappa \leq 4$, D is a domain with smooth boundary, and we try to define the limiting measures as in the first section. However, instead of dividing by Z_N we will multiply by N^{2b}. If $Z_N \sim cN^{-2b}$ as we expect (where the constant c depends on the domain and boundary points, and we ignore the serious lattice effects at the boundary), then the limiting measure will be a nonzero finite measure on paths that is not necessarily a probability measure. Let us call this measure $m(D; z, w)$. Now suppose $D_1 \subset D$ with the perturbation being away from z, w. Then the measure $m(D_1; z, w)$ is absolutely continuous with respect to $m(D; z, w)$. In fact, we can give the Radon-Nikodym derivative

$$\frac{dm(D_1; z, w)}{dm(D; z, w)}(\gamma) = e^{(\mathbf{c}/2)\,\Lambda(D; D_1, \gamma)}\, 1\{\gamma \subset D_1\}, \tag{6.1}$$

where $\Lambda(D; D_1, \gamma)$ is a conformal invariant (independent of κ) that roughly gives the measure of the set of Brownian loops in D that intersect both γ and D_1. This measure was introduced in [31].

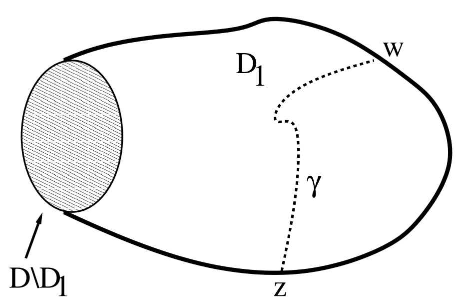

FIGURE 11. Boundary perturbation

Examples.

- The discrete measure on self-avoiding walks gives all walks of the same length the same measure. Hence in a scaling limit, one would expect that boundary perturbation would not affect the measure, i.e., that the limit of SAWs should have central charge $\mathbf{c} = 0$. This property is called the *restriction property*. Assuming that the limit lies on simple curves, we see that the only possible value is $\kappa = 8/3$.
- In the discrete measure on a loop-erased random walk, perturbing the domain *does* affect the measure of a path. The measure of a particular walk

in the LERW is the measure of all the simple random walks whose loop-erasure gives that walk. If we shrink the domain, we lose some of the simple walks that produce our walk and hence the measure decreases. Hence we would expect that $\mathbf{c} < 0$. We have already given a reason why $\kappa = 2$ should correspond to LERW which would give $\mathbf{c} = -2$. On the discrete level one can give a precise description of how the measure changes in terms of random walk loops that hit both the path and the part of the domain being removed. The Brownian loop measure is a continuous limit of the corresponding measure on random walks.

The description (6.1) in terms of the central charge as an exponent in a Radon-Nikodym derivative can be considered a global version of infinitesimal calculations done in [28]. Such infinitesimal calculations also exist in nonrigorous conformal field theory; see, e.g., [16, Section 9.1]. In fact, many of the calculations of SLE repeat computations done in the physics literature. However, the SLE computations have the advantage not only of *rigor* but also of *precision*. In other words, mathematical quantities are defined exactly before they are calculated or estimated, and the relationship between the calculations and the geometric properties of curves and interfaces is established rigorously.

7. Particular models

7.1. **Schramm's original paper.** In [36], Oded Schramm considered two of the models we discussed: LERW and percolation. Assuming conformal invariance and the domain Markov property, he deduced that *if the processes had a conformally invariant limit*, then they must be SLE_κ for some κ. He also did some computations for SLE to determine what the values must be. For LERW he determined $\kappa = 2$ by some winding number calculations and for percolation he determined $\kappa = 6$ by proving an analogue of Cardy's formula for SLE and showing which one matched up appropriately.

7.2. **Brownian paths.** There are a number of problems about the geometric and fractal properties of planar Brownian motion which have been solved using SLE. Benoit Mandelbrot [34] looked at "Brownian islands" which are formed by taking planar Brownian motions, conditioning them to begin and end at the same point, and then filling in the set surrounded by the path. The "coastline" or "outer boundary" of this set is an interesting fractal that appeared to him (and was tested by numerical simulation) to be of fractal dimension 4/3. Since this was also the conjectured dimension for the scaling limit of a self-avoiding walk, he proposed that the outer boundary of Brownian motion might give a model of the limit of SAWs. The dimension of this coastline can be shown to be given in terms of a particular value of the *intersection exponents* for Brownian motion. Wendelin Werner and I [29, 30] were trying to compute the intersection exponents and had discovered, at least heuristically, that these exponents were related to those for SAW and critical percolation. This work was going on at the same time that Schramm was introducing his new process as a model for scaling limits of LERW and percolation. The three of us joined forces to see if SLE could be applied to the intersection exponent problem. This program turned out to be successful [23, 24, 25], and we proved Mandelbrot's conjecture. In the process, we discovered an important property, *locality*, that distinguishes $\kappa = 6$ from all values. Although SLE_6 and

Brownian motion are very different processes, their outer boundaries are the same and are versions of $SLE_{8/3}$. This is the duality relation for $\mathbf{c} = 0$.

7.3. **Percolation.** When Cardy [7, 8] gave his original prediction for the crossing probabilities for critical percolation, he gave it in the upper half-plane $\mathbb{H}$ for which it is a hypergeometric function. Lennart Carleson made the observation that the formula was nicer if one maps on the equilateral triangle, and he also suggested that it might be easier to prove limit theorems for the discrete model on the triangular lattice rather than the square lattice. Indeed, Schramm's percolation exploration process is much easier to define on the triangular lattice. Stas Smirnov [40] proved that the scaling limit of percolation satisfies Cardy's formula, and in the process he established that the percolation exploration process approaches SLE_6. The scaling limit of percolation is much richer than just one boundary curve; however (see [6]), one can use an infinite collection of boundary curves to construct the scaling limit. Smirnov's proof unfortunately holds only for the triangular lattice, and it is still open to prove Cardy's formula for other lattices such as the square lattice. The percolation exploration process satisfies a locality property, inasmuch as to determine an initial segment of the path one needs only examine the colors of the sites adjacent to the path. In particular, if the domain were perturbed away from the path, this would not affect the distribution. The SLE_6 locality property is a continuous analogue of this. Although the discrete percolation exploration process has no self-intersections, its scaling limit, SLE_6, does not have simple paths.

7.4. **Loop-erased random walk.** The loop-erased random walk is related to a number of models, in particular the problem of choosing a spanning tree at random from the collection of all spanning trees on a graph (or, similarly, counting the number of spanning trees); see, e.g., [43]. Roughly speaking, the path connecting two points on a uniform spanning tree has the distribution of a LERW. The spanning tree problem is also related to dimer configurations. Kenyon [19] used this relation to show that the average number of steps in a LERW on an $N \times N$ box is $N^{5/4}$. The process is also a determinantal process and this combined with conformal invariance of Brownian motion can establish conformal invariance for some quantities without discussing *SLE* [14, 20]. In [26], LSW established that the scaling limit of loop-erased random walk is SLE_2 and also the scaling limit of uniform spanning trees (appropriately defined) gives SLE_8. This is another example of the duality relation, this time for $\mathbf{c} = -2$. While there are technical difficulties in the limit, one reason why the problem is tractable is that the LERW and the uniform spanning tree are constructed from a simple random walk and it is well known that the limit of a simple random walk is Brownian motion. The determination of κ comes from the calculation.

7.5. **Gaussian free field.** The Gaussian free field in two dimensions is another fundamental conformally invariant object. We will not define it here but rather recommend [39] for an introduction. Schramm and Sheffield [37, 38] have shown that scaling limits for interfaces for the Gaussian free field (as well as for a somewhat simpler model, the harmonic explorer) are SLE_4 curves. This agrees with what was already known that the free field was a $\mathbf{c} = 1$ model. The proofs of the scaling limit are quite involved. However, as in the case of LERW, the discrete model can be described in terms of random walks and one uses the convergence of a random walk to Brownian motion to establish the result.

7.6. **Ising model and related models.** The interfaces for the Ising model are predicted to converge to SLE_3 curves. There is current exciting work by Smirnov establishing this limit as well as giving a general procedure for determining limits for Potts models and random cluster models (for which the Ising model is an example). An introduction to this work can be found in his expository paper [41].

7.7. **Self-avoiding walk.** If a scaling limit for a self-avoiding walk exists, then it must satisfy the restriction property. This leads us to predict that the scaling limit is $SLE_{8/3}$, and numerical simulations [17, 18] give strong evidence that the conjecture is correct. However, there is no proof that the scaling limit exists.

8. Comments

I will end by making a number of comments about SLE and current and future research trends.

- The relation between conformal invariance and two-dimensional critical phenomena has been studied extensively in the physics community; see, e.g., [9, 15, 11]. For many mathematicians (this author included), this research has been hard to read not just because mathematical details were missing but because the basic notions were not defined precisely enough (for us mortal mathematicians). Many of these ideas can be made precise with SLE—in fact, there have been a number of papers (see, e.g., [1, 10]) that have used SLE to bridge the communication gap between the mathematics and physics communities. There is still much work to be done but one could hope that much of the work of the physics community will be able to be made rigorous using SLE. I should point out that much of the work in the nonrigorous treatments of conformal field theory and relation to critical phenomena is either rigorous or relatively easily "rigorizable".
- Understanding SLE in nonsimply connected domains and more general Riemann surfaces is an ongoing project. The beauty of Schramm's construction is that conformal invariance and the domain Markov property are sufficient to characterize SLE in all simply connected domains. Nonsimply connected domains D have the property that if one slits the domain (say by taking $D \setminus \gamma(0,t]$), then the slit domain is not conformally equivalent to the original domain. To understand such processes, one probably needs to consider measures on paths that are not probability measures. In most cases one expects these measures to be (at least locally) absolutely continuous with respect to SLE in simply connected domains.
- SLE by definition is a random process that moves in a domain with a moving boundary. However, the processes which are modeled by SLE are not naturally built in time by the formation of an SLE path. Instead, the configurations are weighted by an equilibrium measure, and the "dynamics" of an SLE are really just a description of how a conditional probability or conditional expectation changes as we observe more of the path. The fact that SLE gives a dynamic rather than an equilibrium view of systems has made some things harder to prove than they really should be. Zhan [44] recently gave a nice proof of the reversibility of SLE paths for $\kappa \leq 4$; i.e., if γ is a chordal SLE_κ path from z to w in D, then the time reversal of this path has the same distribution as an SLE_κ from w to z in D.

- This paper has focused on only one major aspect of recent work in conformal invariance and two-dimensional statistical mechanics: the geometric limiting curves given by SLE and random walk loops. There is also a large amount of exciting work in related areas: combinatorics, random matrices, integrable systems, complex and algebraic geometry. To even touch on these areas briefly would make this paper too long. The theory of two-dimensional critical phenomena is a very rich area that combines many different areas of mathematics.
- One of the major limitations of the theory of conformally invariant systems and critical phenomena is that it works only in two dimensions. There are many open, and probably significantly harder, problems in understanding processes in three dimensions. In fact, numerical simulations suggest that the answers may not be nice. For example, the critical exponents in two dimensions turn out to be rational numbers but there is no reason to believe that the corresponding exponents (e.g., γ and ν for the self-avoiding walk) are rational numbers.
- For a more extensive survey of SLE and its relation to discrete models, see [42]. For a mathematical treatment of the Loewner equation and SLE with less discussion of the relevant discrete models, see [21].

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