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Degenerate diffusions, by Panagiota Daskalopoulos and Carlos E. Kenig, EMS Tracts in Mathematics 1, European Mathematical Society, Zurich, 2007, x+198 pp., ISBN 978-3-03719-033-36

The prototypical degenerate diffusion equation is the porous medium equation (PME)

$$\partial_t u = \Delta(u^m),$$

where $u(x,t) \geq 0$, $m > 1$ is a constant, and Δ is the Laplace operator in $\mathbf{R}^d$ for $d \geq 1$. Since

$$\Delta(u^m) = \nabla \cdot (mu^{m-1}\nabla(u)),$$

the quantity mu^{m-1} plays the role of the diffusion coefficient. The PME is parabolic for $u > 0$, but degenerates at $u = 0$ since the diffusion coefficient vanishes there. The earliest occurrence of the PME in the scientific literature seems to be in 1903–04 in the work of J. Boussinesq on ground water flow [5]. In an approximate theory of almost horizontal flow, he obtained the PME with $m = 2$ for the height of the water mound. Subsequently, in the 1930s the petroleum engineers L. S. Leibenzon [9] and M. Muskat [10] independently derived the PME for the evolution of the density of an ideal gas flowing isentropically in a homogeneous porous medium. This flow is characterized by the ideal gas law, the conservation of mass, and an empirical relationship between the pressure and the velocity known as Darcy's law. Solving these equations for the density and scaling out the constants leads to the PME with $m = 1 + \gamma$, where $\gamma \geq 1$ is the exponent in the ideal gas law. In both the ground water and ideal gas flows $m \geq 2$, but various values of $m > 1$ occur in other applications. Much more information can be found in Vazquez's monograph [14].

One of the most striking manifestations of the degeneracy of the PME is the finite speed of propagation of disturbances from rest. Let $u(x, t; u_0)$ denote the solution to the Cauchy (initial value) problem

$$\begin{aligned} \partial_t u &= \Delta(u^m) \text{ in } \mathbf{R}^d \times \mathbf{R}^+, \\ u(x, 0; u_0) &= u_0(x) \text{ in } \mathbf{R}^d. \end{aligned}$$

If u_0 has compact support, then $u(\cdot, t; u_0)$ will be compactly supported for all $t > 0$. For $m = 1$, the PME is the classical equation of heat conduction for which the speed of propagation of disturbances from rest is infinite. In particular, even if the nonnegative initial function has compact support, the solution to the Cauchy problem for the heat equation will be everywhere positive for all $t > 0$.

If we allow $0 < m < 1$ in the PME, the equation is still degenerate at $u = 0$. However, in this case, instead of vanishing, the diffusion coefficient blows up at $u = 0$. Moreover, in this case as for the heat equation, there is an infinite speed of propagation. The case $m > 1$ is often referred to as *slow diffusion* while the case $0 < m < 1$ is called *fast diffusion*. Fast diffusion arises in various plasma physics models as well as in other applications (cf. [14]). There is a further distinction in

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the class of fast diffusions. Let $m_c = (d-2)_+/d$, where $(\cdot)_+ = \max(0,\cdot)$. If $m \geq m_c$, then the PME conserves mass; i.e., if

$$\int_{\mathbf{R}^d} u_0(x)dx = M < \infty,$$

then

$$\int_{\mathbf{R}^d} u(x,t;u_0)dx = M$$

for all $t > 0$. This is no longer the case for $m < m_c$, and we distinguish between *super*- and *subcritical fast diffusion*. In the subcritical regime there is extinction in finite time. If u_0 is bounded and not identically zero, then there exists a $0 < T(u_0) < \infty$ such that $u(x,t;u_0) \equiv 0$ in $\mathbf{R}^d \times (T(u_0),\infty)$. One can also consider *ultrafast diffusion* with $m < 0$. To deal with ultrafast diffusion we rescale the PME as

$$\partial_t u = \Delta\left(\frac{1}{m}u^m\right) = \nabla\cdot(u^{m-1}\nabla u)$$

to retain the (formal) parabolicity. In this form we can also take the limit as $m \to 0$ to obtain the equation $\partial_t u = \Delta(\ln u)$ which arises in the study of Ricci flows (see [13]). The theory for slow and supercritical fast diffusion is quite complete. For subcritical, and even more so, ultrafast diffusion there remain many open problems (cf. [13], [14]).

We can also drop the restriction $u > 0$ by rewriting the PME in the form $\partial_t u = \Delta(|u|^{m-1}u)$. More generally, we can consider the generalized porous medium equation (GPME)

$$\partial_t u = \Delta(\varphi(u))$$

for suitable classes of functions φ. The basic question is what are the assumptions on φ needed to generalize results for the PME to the GPME. One such body of results for the PME is the initial trace theory for continuous weak solutions which we now describe.

Assume $m > 1$. Let B_r denote the ball of radius r centered at $0 \in \mathbf{R}^d$, and let $S_T = \mathbf{R}^d \times (0,T]$. A function $u(x,t)$ is said to be a *continuous weak solution* to the PME in S_T if it is continuous and nonnegative in S_T, and it satisfies

$$\iint_{\mathbf{R}^d\times(\tau_1,\tau_2)} (u^m\Delta\eta + u\partial_t\eta)\,dxdt = \int_{\mathbf{R}^d} u\eta dx\,\Big|_{\tau_1}^{\tau_2}$$

for all τ_j such that $0 < \tau_1 < \tau_2 \leq T$ and for all $\eta \in C^{2,1}(S_T)$ such that $\eta(\cdot,t)$ has compact support for all $t \in [\tau_1,\tau_2]$. It is shown in [2] that corresponding to every continuous weak solution u in S_T there is a unique nonnegative Borel measure ρ on $\mathbf{R}^d$ such that

$$\lim_{t\searrow 0}\int_{\mathbf{R}^d} u(x,t)\eta(x)dx = \int_{\mathbf{R}^d}\eta(x)\rho(dx)$$

for all test functions $\eta \in C_0(\mathbf{R}^d)$. The measure ρ is called the *initial trace of* u and it satisfies a growth condition which limits the amount of mass it can place at infinity. Specifically, there exists a constant $c = c(d,m) > 0$ such that

$$\int_{B_r}\rho(dx) \leq c\left\{\left(\frac{r^\kappa}{T}\right)^{\frac{1}{m-1}} + T^{\frac{d}{2}}u(0,T)^{\frac{\kappa}{2}}\right\},$$

where $\kappa = 2 + d(m-1)$. Roughly speaking, this means that, on average, "$u(x,0)$" cannot grow faster than $|x|^{\frac{2}{m-1}}$ as $|x| \to \infty$.

It is natural to ask when does a nonnegative Borel measure ρ determine a continuous weak solution to the PME. For any nonnegative measure μ, define

$$|||\mu||| = \sup_{r\geq 1} r^{\frac{-\kappa}{m-1}} \mu(B_r).$$

The initial trace ρ of a continuous weak solution satisfies $|||\rho||| < \infty$. Conversely, if ρ is a nonnegative Borel measure with $|||\rho||| < \infty$, then there exists a continuous weak solution $W[\rho](x,t)$ of the PME in S_T with initial trace ρ for $T = c(d,m)/l^{m-1}$, where $c(d,m) > 0$ is a constant and $l = \lim_{r\to\infty} \sup_{R\geq r} R^{\frac{-\kappa}{m-1}} \rho(B_R)$. Moreover

$$W[\rho](x,t) \leq c(d,m) t^{\frac{-d}{\kappa}} (1+|x|^2)^{\frac{1}{m-1}} |||\rho|||^{\frac{2}{\kappa}}$$

in S_T [4].

Given a continuous weak solution u of the PME, we determine a unique initial trace ρ with $|||\rho||| < \infty$. On the other hand, given the measure ρ, we can construct a continuous weak solution $W[\rho]$ which also has initial trace ρ. Dahlberg and Kenig [6] prove that $W[\rho] = u$. A continuous weak solution u to the PME in S_T possesses various regularity properties [6]. For each $t \in (0,T)$,

$$u(x,t) \leq C(1+|x|^2)^{\frac{1}{m-1}} t^{\frac{-d}{\kappa}},$$

where C is a constant which depends only on $T, u(0,T), d$, and m. Moreover, u is Hölder continuous with exponent depending only on d and m on all compact sets $K \subset \mathbf{R}^d \times (0,T)$ and satisfies the Aronson–Bénilan estimates [1]

$$\Delta v \geq \frac{-d}{\kappa t} \quad \text{and} \quad \partial_t v \geq -\frac{(m-1)dv}{\kappa t} \quad \text{in } \mathcal{D}'(S_T),$$

where $v = \frac{m}{m-1} u^{m-1}$. The estimate for Δv is sharp, and equality is achieved for the self-similar explicit Barenblatt solution [3]. If u represents the density of the gas, then, in view of the ideal gas law, v is its scaled pressure.

These results for the slow diffusion case $m > 1$ extend the classical Widder theory [15], which completely characterizes nonnegative solutions to the heat conduction equation. The situation for the fast diffusion case $0 < m < 1$ is radically different. For example, Herrero and Pierre [8] study the Cauchy problem in $\mathbf{R}^d \times \mathbf{R}^+$ with $0 < m < 1$. They prove that if $u_0 \in L^1_{\text{loc}}(\mathbf{R}^d)$, then the Cauchy problem has a time-global continuous weak solution, i.e., they obtain global existence without any growth conditions at infinity. Moreover, for all $t > 0$ and all $R > 0$,

$$\int_{B_R} |u(x,t;u_0)|\, dx \leq C \left\{ \int_{B_{2R}} |u_0|\, dx + t^{\frac{1}{1-m}} R^{\frac{-\kappa}{1-m}} \right\},$$

where C is a constant depending only on d and m. If $m_c < m < 1$ and $u_0 \in L^1_{\text{loc}}(\mathbf{R}^d)$, then there exists a solution to the Cauchy problem such that for all $t > 0$ and all $R > 0$

$$\sup_{x\in B_R} |u(x,t;u_0)| \leq C \left\{ t^{\frac{-\kappa}{d}} \left[\int_{B_{4R}} |u_0|\, dx \right]^{\frac{2}{\kappa}} + \left(\frac{t}{R^2} \right)^{\frac{1}{1-m}} \right\}$$

($L^\infty - L^1$ regularization). More information on the fast diffusion case can be found in [13].

A major portion of the Daskalopoulos–Kenig book is devoted to extending the above results to a suitable class of GPME's. In order to define a suitable class, it is necessary to introduce some structure assumptions on the function $\varphi(u)$. A nonnegative continuous function $\varphi : [0,\infty) \to \mathbf{R}$, which is differentiable on $\mathbf{R}^+$ and normalized by $\varphi(0) = 0$ and $\varphi(1) = 1$, is said to be of class Γ_a if there exists a constant $a \in (0,1)$ such that for any $u > 0$

$$a \leq \frac{u\varphi'(u)}{\varphi(u)} \leq \frac{1}{a}.$$

Thus, functions in Γ_a have power growth at $u = 0$. This is the minimal structure assumption. To extend the results for the slow diffusion case $m > 1$, the authors introduce the subclass $\mathcal{S}_a \subset \Gamma_a$ of functions which are strictly increasing and satisfy the superlinearity condition

$$\frac{u\varphi'(u)}{\varphi(u)} \geq 1 + a$$

for all $u \geq 1$. To extend the results for supercritical fast diffusion $m_c < m < 1$, they introduce the subclass $\mathcal{F}_a \subset \Gamma_a$ of functions which are strictly increasing and satisfy the sublinearity condition

$$\frac{d-2}{d} + a \leq \frac{u\varphi'(u)}{\varphi(u)} \leq 1 - a$$

for all $u \geq 1$. The lower bound $\frac{d-2}{d} + a$ turns out to be essential for $L^\infty - L^1$ regularization.

In the first chapter the authors assume at the very least that $\varphi \in \Gamma_a$, and they develop some of the basic tools they will employ throughout the book. A standard procedure in the study of a degenerate problem is to approximate by nondegenerate problems and seek estimates that allow for passage to the limit. For this purpose one needs a comparison principle as well as some sort of compactness theorem. Here the comparison principle is essentially standard, and the compactness theorem is the equicontinuity result of P. Sacks [12], the detailed proof of which is given in this chapter. There are local L^∞ bounds ($L^\infty - L^1$ regularization) for smooth nonnegative solutions for both $\varphi \in \mathcal{S}_a$ and $\varphi \in \mathcal{F}_a$. Moreover, there is a Harnack type estimate for nonnegative weak solutions when $\varphi \in \mathcal{S}_a$ with $\varphi(u)/u$ monotone increasing on $[1, \infty)$. The chapter ends with an existence proof for a weak solution of the homogeneous initial-Dirichlet problem

$$\begin{aligned} \partial_t u &= \Delta\varphi(u) \text{ in } \Omega \times (0, T], \\ u(x,t) &= 0 \text{ on } \partial\Omega \times (0,T],\ u(x,0) = u_0(x) \text{ on } \Omega, \end{aligned}$$

where Ω is bounded and $u_0 \in L^\infty(\Omega)$, and for the corresponding Cauchy problem with $u_0 \in L^\infty(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$.

In the next chapter the authors extend the results on the initial trace theory outlined above for the PME in the slow diffusion case $m > 1$ to the GPME with $\varphi \in \mathcal{S}_a$. Actually, with the exception of the pointwise estimates and the Pierre uniqueness theorem [11], the detailed proofs are given for the PME and the reader is referred to the literature for the generalizations to the GPME. The chapter concludes with remarks on various complementary topics mainly for the PME.

Chapter 3 consists of three distinct sections. The first is devoted to extending the results of Herrero–Pierre for the PME with $m_c < m < 1$ to the GPME with $\varphi \in \mathcal{F}_a$. In particular, if $\varphi \in \mathcal{F}_a$, then a locally finite Borel measure μ determines a unique continuous weak solution to the GPME with μ as its initial trace. The second section of Chapter 3 concerns the logarithmic diffusion equation (LDE)

$$\partial_t u = \Delta(\ln u)$$

in dimensions $d \geq 2$. As noted above this equation arises as the limit as $m \to 0$ of the scaled PME with $\varphi = \frac{1}{m}u^m$. In the critical dimension $d = 2$, the LDE represents the evolution of the conformally equivalent metric g with $ds^2 = u(dx^2 + dy^2)$ under the Ricci flow. For $d \geq 3$ and radially symmetric locally integrable u_0, there is a growth condition which is necessary and sufficient for the existence of a solution to the Cauchy problem in $\mathbf{R}^d \times (0, T)$. A counterexample shows that this condition does not guarantee existence for nonradially symmetric u_0. For $d = 2$, there is a corresponding growth condition which is necessary and sufficient for existence regardless of the symmetry of u_0. There is also a strong nonuniqueness property. For given u_0 and for every $s \in [0, \infty)$, there exists a solution $u_s(x, t; u_0)$ to the Cauchy problem in $\mathbf{R}^d \times (0, T_s)$ where

$$T_s = \frac{1}{2\pi(2+s)} \int_{\mathbf{R}^2} u_0(x)dx,$$

such that $u_s(x, t; u_0) = 0$ for all $t \geq T_s$. The final section of Chapter 3 summarizes various results on the time-asymptotic behavior and the existence or nonexistence of solutions to the Cauchy problem for the PME in the fast and ultrafast diffusion regimes. For subcritical fast diffusion and ultrafast diffusion there is extinction in finite time. Moreover, for ultrafast diffusion the solution can fail to exist since extinction is instantaneous if the initial data u_0 decays too rapidly at infinity.

Chapter 4 deals with the homogeneous initial Dirichlet problem in an infinite cylinder $\Omega = D \times \mathbf{R}^+$, where $D \subset \mathbf{R}^d$ is an open bounded set with smooth boundary. If $\varphi \in \mathcal{S}_a$, then this problem has a maximal strong (i.e., continuous) solution $\alpha(x, t)$, called the "friendly giant", which has an infinite trace at $t = 0$, and which is such that for any other strong solution u we have $u \leq \alpha$. If $\lim_{h \searrow 0} \frac{\varphi(hu)}{\varphi(h)}$ exists uniformly on compact subsets of $[0, \infty)$, then α attracts all strong solutions as $t \to \infty$. For any solution u other than α there exist nonnegative Borel measures μ on D and λ on ∂D satisfying appropriate finiteness conditions such that for any $\eta \in C_0^\infty(\mathbf{R}^d)$ with $\eta|_{\partial D} = 0$,

$$\lim_{t \searrow 0} \int_D u(x.t)\eta(x)dx = \int_D \eta(x)\mu(dx) - \int_{\partial D} \partial_n \eta(\sigma)\lambda(d\sigma).$$

Strong solutions are uniquely determined by their initial trace, and any given Borel measures λ, μ satisfying the finiteness conditions determine a unique strong solution with these measures as initial trace. In the fast diffusion case for the PME, there are analogous trace and uniqueness results, but the existence result holds only for $\frac{(d-1)_+}{d} < m < 1$.

The final chapter is devoted to proving that every distribution solution to the PME with $m > 1$ defined in an open set $\Omega \subset \mathbf{R}^d \times \mathbf{R}$ with $u \in L^m_{\mathrm{loc}}(\mathbf{R}^d \times \mathbf{R})$ has a representative which is continuous a.e. in Ω. The same result holds for the GPME if $\varphi \in \mathcal{S}_a$ and is convex. Whether this remains true without convexity is an open question.

This is not a book for beginners. The reader will need a considerable degree of mathematical sophistication. A working knowledge of real analysis and partial differential equations is called for at a minimum. There are no exercises, but there are various open research problems cited in the notes at the end of chapters. Advanced graduate students and experienced researchers will find much to interest them here. The proofs are concise and elegant, and some are quite novel. There is considerable overlap with the monographs of Vazquez [13], [14] which present a comprehensive theory of the PME. Where they do overlap, the treatments are sufficiently distinct so that the interested reader will learn a great deal by consulting both Daskalopoulos–Kenig and Vazquez. There is essentially no overlap with the similarly titled book by DiBenedetto [7] which treats the degenerate parabolic equation $\partial_t u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ for $p > 1$.

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