

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by

HERWIG HAUSER

MR0199184 (33 #7333) 14.18

Hironaka, Heisuke

Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.

Annals of Mathematics. Second Series **79** (1964), 109–203; *ibid.* (2) **79** (1964), 205–326.

Outre la solution d'un problème célèbre, ce travail donne un des outils les plus puissants de la géométrie algébrique. Ce résultat est atteint avec une surprenante économie de moyens (notion de platitude, structure des anneaux locaux complets, fermeture du lieu singulier), compensée par une étonnante habileté. L'auteur utilise le langage des schémas, et c'est là un point essentiel, mais seulement les notions les plus simples et les plus naturelles: produit fibré, schéma projectif associé à une algèbre graduée. Quant aux résultats, énonçons le plus simple. Soit X un préschéma noethérien réduit, irréductible et "excellent" (p. 161) dont les anneaux locaux sont de caractéristique résiduelle nulle. Soit $U = X - S$, où S est le lieu singulier de X (ensemble des points où l'anneau local n'est pas régulier). Il existe un morphisme surjectif $r: R \rightarrow X$ tel que R soit régulier, $r^{-1}(S)$ étant un diviseur à croisements normaux et r induisant un isomorphisme $r^{-1}(U) \xrightarrow{\sim} U$. On pourra prendre pour X une variété algébrique sur un corps de caractéristique nulle car ses anneaux locaux sont "excellents". Un anneau local complet l'est aussi; on sait donc résoudre les singularités des "variétés algébroides". Il est d'ailleurs notable que le principe même de la démonstration suppose que l'on étudie simultanément les variétés algébriques et les variétés algébroides, ce que permet le langage des schémas. L'auteur obtient également la résolution "locale" des singularités d'un sous-espace analytique complexe de $S \times P$, où S est une variété de Stein et P un espace projectif complexe.

La démonstration se fait par une récurrence subtile qui exige que l'on prouve à chaque cran un résultat plus précis que celui annoncé plus haut; en particulier, on impose que le morphisme résolvant $r: R \rightarrow X$ soit obtenu en composant un nombre fini d'éclatements "permis". Un éclatement permis est un morphisme $X' \rightarrow X$ obtenu en faisant éclater dans X un préschéma régulier D contenu dans le lieu singulier de X et tel que le cône normal de X le long de D soit plat sur D . Pour un tel D , la singularité de X est la même en tous les points de D , en un sens très fort (polynôme de Hilbert). Si X est une hypersurface d'un schéma régulier Y , on voit que D est permis si D est régulier et si X a même multiplicité en tous les points de D (Chapitre II). On prouve ensuite qu'un éclatement permis ne peut que rendre X "moins singulier", le degré de singularité de X en un point $x \in X$ étant mesuré par deux suites d'entiers définies en termes du cône tangent (Chapitre III). Le reste du Chapitre III est consacré au cas où X est le spectre d'un anneau local complet, donné comme quotient d'un anneau local complet régulier R . Il s'agit de trouver des paramètres de R et des équations de X ayant des propriétés très fortes, stables par éclatement permis. Ce résultat de démonstration délicate est fondamental. On

l'utilise comme suit: ayant construit un éclatement permis $p: X' \rightarrow X$ destiné à désingulariser certains points de X , les paramètres et équations évoqués plus haut servent à contrôler les anneaux locaux des points $x' \in X'$ qui sont aussi singuliers que $p(x')$.

La clef du problème consiste alors en une formulation plus précise: les théorèmes A , B , C et D du Chapitre I. Il reste à tirer le feu d'artifice en les démontrant simultanément par une récurrence savante (Chapitre IV).

[A translation of the French text above follows:

Aside from solving a famous problem, this work provides one of the most powerful tools of algebraic geometry. This is achieved via a surprising economy of means (notion of flatness, structure of complete local rings, closure of the singular locus), balanced by stunning technical skill. The author uses the language of schemes—an essential aspect—but only the simplest, most natural notions: pullback, projective scheme associated with a graded algebra. As for the results, let us cite the simplest one. Let X be an “excellent” (p. 161) reduced irreducible Noetherian prescheme whose local rings have residue fields of characteristic zero. Let $U = X - S$, where S is the singular locus of X (the set of points where the local ring is not regular). There exists a surjective morphism $r: R \rightarrow X$ such that R is regular, where $r^{-1}(S)$ is a normal crossing divisor and r induces an isomorphism $r^{-1}(U) \xrightarrow{\sim} U$. One can take X to be an algebraic variety over a field of characteristic zero since its local rings are excellent. Because a complete local ring is also excellent, the singularities of the “algebroid varieties” can be resolved. Moreover, it is notable that the very principle of the proof assumes that the algebraic varieties and the algebroid varieties are studied simultaneously, making it possible to use the language of schemes. The author also obtains the “local” resolution of the singularities of a complex-analytic subspace of $S \times P$, where S is a Stein variety and P a complex projective space.

The proof is carried out via a subtle recurrence that requires proving at each step a more precise result than the one announced above; in particular, the resolution morphism $r: R \rightarrow X$ is required to be obtained by a composition of finitely many “permissible” blowups. A permissible blowup is a morphism $X' \rightarrow X$ obtained by blowing up in X a regular prescheme D contained in the singular locus of X and such that the normal cone of X along D is flat on D . For such a D , the singularity of X is the same at every point of D , in a very strong sense (Hilbert polynomial). If X is a hypersurface of a regular scheme Y , then D is permissible if D is regular and if X has the same multiplicity at every point of D (Chapter II). The author then proves that a permissible blowup can only make X “less singular”, with the degree of singularity of X at a point $x \in X$ being measured by two integer sequences defined in terms of the tangent cone (Chapter III). The rest of Chapter III is devoted to the case in which X is the spectrum of a complete local ring, given as the quotient of a regular complete local ring R . The goal is to find parameters of R and equations for X having very strong properties, stable under a permissible blowup. This result, with its delicate proof, is fundamental. It is used as follows: once a permissible blowup $p: X' \rightarrow X$ designed to desingularize certain points of X is constructed, the above-mentioned parameters and equations serve to control the local rings of the points $x' \in X'$ which are as singular as $p(x')$.

The key to the problem then lies in a more precise formulation: Theorems A , B , C and D of Chapter I. The fireworks start when these theorems are proved simultaneously via a masterly recurrence (Chapter IV).]

{See also [A. Grothendieck, Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 95–103; MR0199194 (33 #7343)].}

From MathSciNet, September 2009

J. Giraud

MR1978567 (2004d:14009) 14E15; 32S45

Hauser, Herwig

The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand).

Bulletin of the American Mathematical Society (New Series) **40** (2003), no. 3, 323–403 (electronic).

This paper is expository in nature. It seduces the reader with the charming flavour and lightness of a bedtime story told to children: it gets the audience gradually involved in and obsessed with the theme (in the case of children, until they fall asleep; in the case of mathematicians, until they pick up paper and pencil to solve the proposed riddles on their own). In its size and organization the paper has the status of a monograph. It explains how to prove the existence of resolutions of singularities of algebraic varieties over a field of characteristic zero. Many mathematicians have been fascinated by this problem; see the introduction of [*Resolution of singularities (Obergrugl, 1997)*, Progr. Math., 181, Birkhäuser, Basel, 2000; MR1748614 (2000k:14002)]. In [Ann. of Math. (2) **79** (1964), 109–203; *ibid.* (2) **79** (1964), 205–326; MR0199184 (33 #7333)] H. Hironaka proved the existence of resolution of singularities in characteristic zero; this was the first result for varieties of any dimension, but the theorem is non-constructive. At the end of the last century the constructiveness of Hironaka's theorem was proved [see, e.g., O. E. Villamayor U., Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32; MR0985852 (90b:14014); E. Bierstone and P. D. Milman, Invent. Math. **128** (1997), no. 2, 207–302; MR1440306 (98e:14010)], and people also got interested in the method for resolving singularities, not only in the existence.

Given an algebraic variety X over a field of characteristic zero and embedded in a regular variety W , an embedded resolution of singularities of $X \subset W$ is a proper and birational morphism $W' \rightarrow W$ such that W' is regular, the strict transform X' of X has no singular points and the total exceptional divisor of the morphism has only normal crossings with X' . Let us recall that the strict transform X' may be defined as the closure of the inverse image of regular points of X , and that a scheme E is said to have only normal crossings at a point if there is a regular system of parameters (say a system of coordinates) at the point x_1, \dots, x_n such that E may be expressed as the zero set of monomials on the x_i 's. It is very natural to require in addition the property that the morphism $W' \rightarrow W$ is an isomorphism outside the singular points of X and other properties like equivariance under group actions; also, usually, the morphism $W' \rightarrow W$ is required to be a sequence of blowups at regular centers. So the problem is to define the several centers to be blown up.

All constructive proofs define an upper semicontinuous (u.s.c.) function on W such that the points where the function is maximum form the first center. One obtains the first blowup $W_1 \rightarrow W$; then a u.s.c. function is constructed on W_1 defining $W_2 \rightarrow W_1$, and the procedure continues until the resolution is achieved. The proof of termination follows by the improvement of the function, namely the

function decreases at each stage. The goal of the paper is to construct those functions which define the sequence of centers of the blowups. But, unlike in the usual mathematical research papers, the author proceeds in a different way: after describing the problem he develops a naive strategy to try to solve it. This strategy immediately hits obstructions. Studying these, the author (as well as the reader) is led to modify and improve the strategy stepwise, exploring and thus discovering the (rather complicated) structure of the final proof. As the author says, the reader develops his own proof (under the auspices of the guide).

In the paper all concepts and ideas are introduced for non-specialists (at least at the beginning). There are three chapters: Chapter 0 for busy readers, Chapter 1 for moderately interested readers and Chapter 2 for highly interested readers. Thus, everyone can read only up to the chapter which best fits his interest. But it could happen that after Chapter 0 the reader may already be so fascinated that he is tempted to continue with Chapter 1 and also with the more technical Chapter 2. As the author says, “The question is: How can I understand in one hour the main aspects of a proof which originally covered two hundred pages?” The objective of the article under review is to reveal to the reader the beauty of the problem and to explain to him the main points of the proof. At the end of the introduction we find the sentence: “The article has accomplished its goal if the reader starts to suspect—after having gone through the complex and beautiful building Hironaka proposes—that he himself could have proven the result, if only he had known that he was capable of it.” And the author may have succeeded in this goal.

In the paper there are several pictures illustrating abstract concepts, and also lots of examples are found everywhere, helping to motivate the necessity of defining new objects.

Chapter 0 is an overview of the problem of resolution of singularities, written for non-specialists. First it introduces one of the main concepts, the blowup; then there is a dictionary which translates concepts from algebraic geometry (high-tech) to intuitive concepts (low-tech). Followed by an explanation of the result, and a brief exposition of the inductive nature of the proof, several examples illustrate concepts like: the strict and weak transform of an ideal, the coefficient ideal of an ideal and the construction of the centers of the blowups which make the singularities improve and finally define a resolution of singularities.

Chapter 1 starts with the main ideas of Hironaka’s proof, which may be summarized as follows: how to choose the center at each stage of the resolution process in order to improve the singularities of X . Here the idea of the coefficient ideal is developed in more detail. At every point, how the coefficient ideal should be defined and how it transforms under blowup is justified with examples. The author concludes by the definition of an invariant which will work: it is well defined (does not depend on any choices), and it has the properties required at the beginning. But immediately some obstructions are explained, in order to have an invariant which bring us to the end. In particular one of the obstructions is the normal crossings condition on the center with respect to the exceptional divisor. Thus the invariant must be refined or transformed. At the end of Chapter 1 the author introduces us to the positive characteristic problem and explains the first obstacles in this case.

Chapter 2 is devoted to giving the precise definitions and proofs to the construction of resolution of singularities in characteristic zero. The use of mobiles has to be mentioned. Mobiles are objects attached to the singularity of X which were introduced in [S. Encinas and H. Hauser, *Comment. Math. Helv.* **77** (2002), no. 4,

821–845 MR2004c:14021]. In the literature one finds several objects encoding the data for resolving singularities: Hironaka’s idealistic exponents; Abhyankar’s trios, quartets and quintets; Villamayor’s basic objects; and Bierstone–Milman’s infinitesimal presentations. Mobiles seem to be the final concept for how to encode the required resolution data. In contrast to the earlier concepts, they are intrinsic and they collect the precise information which one wishes to deal with at each stage of the resolution process. Using mobiles eliminates the need to consider the history of prior blowups and makes equivalence relations superfluous.

At the end of Chapter 2 the author deals with the problems in positive characteristic. Two examples are developed illustrating how the characteristic-zero techniques fail in several aspects, all motivated by the non-existence of maximal contact hypersurfaces in positive characteristic.

The paper terminates with five appendices: Appendices A, B and C explain technical details of the proof of the theorem. Appendices D and E are a resumé of definitions and a table of notations useful to follow the proofs, especially in Chapter 2.

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Santiago Encinas

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Cutkosky, Steven Dale

Resolution of singularities. (English)

Graduate Studies in Mathematics, 63.

American Mathematical Society, Providence, RI, 2004. viii+186 pp.

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Resolution of singularities is a classical (and difficult) problem in algebraic geometry, not only interesting in itself but also with important applications.

For a long time, desingularization was known to exist only for curves. In the second third of the last century Walker, Zariski and Abhyankar published the first proofs in higher dimensions [R. J. Walker, *Ann. of Math.* (2) **36** (1935), no. 2, 336–365; MR1503227; O. Zariski, *Ann. of Math.* (2) **40** (1939), 639–689; MR0000159 (1,26d); *Ann. of Math.* (2) **43** (1942), 583–593; MR0006851 (4,52c); *Ann. of Math.* (2) **45** (1944), 472–542; MR0011006 (6,102f); S. Abhyankar, *Ann. of Math.* (2) **63** (1956), 491–526; MR0078017 (17,1134d)]; see also [O. Zariski, *Ann. of Math.* (2) **41** (1940), 852–896; MR0002864 (2,124a)] for a local result in arbitrary dimensions.

In the mid 1960s H. Hironaka proved his celebrated result, showing the existence of desingularization for algebraic varieties over fields of characteristic zero [*Ann. of Math.* (2) **79** (1964), 109–203; *ibid.* (2) **79** (1964), 205–326; MR0199184 (33 #7333)]. Since then, the positive characteristic case has resisted the attack of experts; partial results were due to Abhyankar, who succeeded in proving the theorem for surfaces, and if the characteristic is larger than five, for three-folds [op. cit.; *Ramification theoretic methods in algebraic geometry*, *Ann. of Math. Stud.*, 43, Princeton Univ. Press, Princeton, N.J., 1959; MR0105416 (21 #4158); *Resolution of singularities of embedded algebraic surfaces*, Second edition, Springer, Berlin, 1998; MR1617523 (99c:14021)]; and there was also J. Lipman’s theorem, valid for arbitrary excellent surfaces [*Ann. of Math.* (2) **107** (1978), no. 1, 151–207; MR0491722 (58 #10924)].

During the 1980s and 1990s the first algorithmic proofs of desingularization were published [O. E. Villamayor U., *Ann. Sci. École Norm. Sup. (4)* **22** (1989), no. 1, 1–32; MR0985852 (90b:14014); *Ann. Sci. École Norm. Sup. (4)* **25** (1992), no. 6, 629–677; MR1198092 (93m:14012); E. Bierstone and P. D. Milman, *Invent. Math.* **128** (1997), no. 2, 207–302; MR1440306 (98e:14010); S. Encinas and O. E. Villamayor U., *Acta Math.* **181** (1998), no. 1, 109–158; MR1654779 (99i:14020); S. Encinas and O. E. Villamayor U., in *Resolution of singularities (Obergrugl, 1997)*, 147–227, *Progr. Math.*, 181, Birkhäuser, Basel, 2000; MR1748620 (2001g:14018)]. All of them were based on Hironaka’s argument, and some of them have been implemented as computer programs (the computer program in [G. Bodnár and J. Schicho, in *Resolution of singularities (Obergrugl, 1997)*, 231–238, *Progr. Math.*, 181, Birkhäuser, Basel, 2000; MR1748621 (2001e:14001)] is based on the algorithm described in [S. Encinas and O. E. Villamayor U., *op. cit.*; MR1748620 (2001g:14018)]; see also G. Bodnár and J. Schicho, *J. Symbolic Comput.* **30** (2000), no. 4, 401–428; MR1784750 (2001i:14083); “desing—A computer program for resolution of singularities”, *Res. Inst. Symb. Comput.*, available at <http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/>; A. Fruehbs-Krueger and G. Pfister, “Singular library for resolution of singularities”; *per revr.*). A conceptually different algorithm appeared in [S. Encinas and O. E. Villamayor U., *Rev. Mat. Iberoamericana* **19** (2003), no. 2, 339–353; MR2023188 (2004m:14017)], where some of the technicalities of Hironaka’s proof were avoided; in [S. Encinas and H. Hauser, *Comment. Math. Helv.* **77** (2002), no. 4, 821–845; MR1949115 (2004c:14021)] there is a significant simplification in the presentation of the inductive data of the algorithm; in [A. Bravo and O. E. Villamayor U., *Proc. London Math. Soc. (3)* **86** (2003), no. 2, 327–357; MR1971154 (2004c:14020)] the output of the algorithm is stronger than Hironaka’s statement [see also A. Bravo and O. E. Villamayor U., *Math. Res. Lett.* **8** (2001), no. 1–2, 79–89; MR1825262 (2002b:14019)], and in [J. Włodarczyk, “Simple Hironaka resolution in characteristic zero”, preprint, arxiv.org/abs/math/0401401] we can find an interesting simplification of the algorithms of resolution [see also E. Bierstone and P. D. Milman, *Mosc. Math. J.* **3** (2003), no. 3, 751–805, 1197; MR2078560; S. S. Abhyankar, *Adv. in Math.* **68** (1988), no. 2, 87–256; MR0934366 (89e:14012)].

In the early 1990s A. J. de Jong came up with significant new ideas, applying the theory of moduli spaces to desingularization. His results led to a simpler (but weaker) proof of desingularization in characteristic zero, and resolution in positive characteristic up to alterations [*Inst. Hautes Études Sci. Publ. Math.* No. 83 (1996), 51–93; MR1423020 (98e:14011)].

While resolution theorems are used quite frequently in algebraic geometry, very few people have mastered all the details of Hironaka’s proof. However, recently there has been an increasing interest in understanding the main ideas behind Hironaka’s resolution process, and the response to it has been a series of publications (for instance [H. Hauser, *Bull. Amer. Math. Soc. (N.S.)* **40** (2003), no. 3, 323–403 (electronic); MR1978567 (2004d:14009); A. Bravo, S. Encinas, O. E. Villamayor U., “A simplified proof of desingularization and applications”, *Rev. Mat. Iberoamericana*, to appear; O. E. Villamayor U., *An introduction to constructive desingularization*, notes in preparation], which try to make these ideas more accessible). The book under review falls within this framework.

Cutkosky's text covers the main classical results in the resolution of singularities, from Puiseux expansions to a detailed exposition of Hironaka's approach, including chapters dedicated to Zariski's results in uniformization, resolution in positive characteristic, and simultaneous resolution.

The starting point is an overview of the main results in desingularization, which gives a global idea of the development of the subject. This is the content of Chapter one. Chapter two summarizes the main concepts: regularity versus smoothness, normalization, Macaulayfication, uniformization. Once the technical language is introduced, resolution type problems are stated.

Resolution of curve singularities is without any question the easiest-to-handle case. Treated in Chapter three, it already contains the main ideas (maximal contact and induction) of the proof in the general case, and has the obvious advantage of being much simpler. Chapter four introduces the basic tools in resolution with all generality: blow-ups of ideals, total and strict transforms, and universal properties of blowing-ups. Chapter five deals with surface singularities. These first chapters are an appropriate antecedent to Chapter six, where a complete proof of resolution of singularities of algebraic varieties over fields of characteristic zero is given, following the algorithm from [S. Encinas and O. E. Villamayor U., op. cit.; MR1748620 (2001g:14018)].

The positive characteristic case is treated in Chapter seven. The main obstacle for extending Hironaka's proof to this case is the failure of maximal contact, which is illustrated with examples and exercises. This chapter contains a proof of desingularization of surfaces in positive characteristic, following Hironaka's notes "Desingularization of excellent surfaces", published in the appendix of [V. Cosart, J. Giraud and U. Orbanz, *Resolution of surface singularities*, Lecture Notes in Math., 1101, Springer, Berlin, 1984; MR0775681 (87e:14032)]. References to Abhyankar's work on surfaces and three-folds are also given.

The first algebraic approaches to resolution of singularities are due to Zariski using valuative criteria. He proved local uniformization of algebraic function fields, and this led him to prove resolution of surfaces and three-folds. Chapter eight is dedicated to describing Zariski's work in [op. cit.; MR0000159 (1,26d); op. cit.; MR0006851 (4,52c)].

Chapter nine deals with simultaneous resolution and ramification of valuations, presenting some of the results of the author in these subjects [cf. Proc. Amer. Math. Soc. **128** (2000), no. 7, 1905–1910; MR1646312 (2000m:14013); S. D. Cutkosky and O. Piltant, Adv. Math. **183** (2004), no. 1, 1–79; MR2038546].

It has been a pleasure for the reviewer to read this beautiful book, which is a must for graduate students interested in the subject. It fills a gap in graduate texts, covering the most important results in resolution of singularities in an elegant and didactic style. It also includes a large list of references to the basic and more relevant material in the area, containing in addition an interesting collection of exercises.

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