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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
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MR1359899 (96m:20019a) 20C30; 20C20, 20G05

Kleshchev, A. S.

Branching rules for modular representations of symmetric groups. I.

J. Algebra **178** (1995), *no.* 2, 493–511.

MR1319521 (96m:20019b) 20C30; 20C20, 20G05

Kleshchev, Alexander S.

Branching rules for modular representations of symmetric groups. II.

J. Reine Angew. Math. **459** (1995), 163–212.

MR1395065 (96m:20019c) 20C30; 20C20, 20G05

Kleshchev, A. S.

Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux.

J. London Math. Soc. (2) **54** (1996), *no.* 1, 25–38.

I am going to refer to the three papers by I, II, III respectively.

This series of papers deals with two problems in the representation theory of symmetric groups. Let us denote the symmetric group on n letters by Σ_n. What happens when we restrict a representation of Σ_n to Σ_{n-1}? (Here we regard Σ_{n-1} as embedded into Σ_n in the obvious way as the stabilizer of one element.) What happens when we take the tensor product of a representation of Σ_n with the one-dimensional sign representation?

Over a field of characteristic 0 the answers to these questions are well known. It suffices to deal with irreducible representations of Σ_n. These are parametrized by the partitions of n, i.e., by the sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0)$ with $\sum_j \lambda_j = n$. Denote the module corresponding to λ by $S(\lambda)$. The characters of $S(\lambda)$ were determined by F. G. Frobenius in 1900. I. Schur found in 1901 another formula from which one easily deduces (cf. his 1908 paper, top of page 253 [*Gesammelte Abhandlungen. Band I*, 251–265, Springer, Berlin, 1973; MR0462891 (57 #2858a)]) that the restriction of $S(\lambda)$ to Σ_{n-1} is the direct sum of all $S(\lambda^{(i)})$ with $\lambda_i > \lambda_{i+1}$. Here $\lambda^{(i)}$ is the partition of $n-1$ that one gets from λ by replacing λ_i by $\lambda_i - 1$. Furthermore, the tensor product of $S(\lambda)$ with the sign representation is isomorphic to $S({}^t\lambda)$, where ${}^t\lambda$ is the "transposed" partition of λ, i.e., the partition whose ith part $({}^t\lambda)_i$ is the number of m with $\lambda_m \geq i$. This result goes back to §6 of Frobenius' 1900 paper [*Gesammelte Abhandlungen. Band III*, 148–166, Springer, Berlin, 1968; MR0235974 (38 #4272)]. At that point Frobenius formulates his result somewhat differently, but §5 in his 1903 paper [op. cit., 244–274] shows the equivalence of the two formulations. (Both $\lambda^{(i)}$ and ${}^t\lambda$ are probably better understood in terms of their Young diagrams: they arise from that of λ by removing a corner node and by reflection about the diagonal, respectively.)

The situation over a field of prime characteristic p is much more complicated. Here the irreducible representations are parametrized by the "p-regular" partitions of n, i.e., by those partitions λ where for each i there are at most $p-1$ indices m with $\lambda_m = i$. Denote the simple module corresponding to λ by $D(\lambda)$. We do not know (in general) the characters of these modules, not even their dimensions [cf. G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, Reading, MA, 1981; MR0644144 (83k:20003)].

It is clear also in characteristic p that there is for each λ a λ' such that $D(\lambda')$ is isomorphic to the tensor product of $D(\lambda)$ with the sign representation. But we cannot expect λ' to be equal to ${}^t\lambda$ in general, since the map $\lambda \mapsto {}^t\lambda$ will not preserve the set of p-regular partitions. In 1979 G. Mullineux [J. London Math. Soc. (2) **20** (1979), no. 1, 60–66; MR0545202 (80j:20016)] described an algorithm and conjectured that it would produce λ' if the input is λ. This conjecture has been checked in several cases; for $n < 3p$ it has been proved by S. Martin [Quart. J. Math. Oxford Ser. (2) **41** (1990), no. 161, 79–92; MR1044757 (91d:20018)]. Now in Section 4 of III Kleshchev describes another algorithm and proves that it does produce λ' if the input is λ. In more recent work ["A proof of the Mullineux conjecture", Math. Z., to appear] B. Ford and Kleshchev have shown that Mullineux's algorithm produces the same partition as Kleshchev's.

Kleshchev's result on tensoring with the sign representation is based on his work on the restriction of $D(\lambda)$ from Σ_n to Σ_{n-1} (see below). It implies that $D(\lambda)$ is determined by its block together with its restriction to Σ_{n-1}. (See III.3.3 for a more precise statement.) This enables Kleshchev to use an inductive construction.

It had been known for some time that restrictions of simple Σ_n-modules to Σ_{n-1} are not semisimple (in general) and that simple Σ_{n-1}-modules can occur as composition factors with arbitrarily large multiplicity. A typical example for this behaviour was found by G. D. James [J. Algebra **43** (1976), no. 1, 45–54; MR0430050 (55 #3057b)]. Kleshchev discovered that the situation gets better if one looks only at the socle of $D(\lambda)$ restricted to Σ_{n-1} and that this socle contains already rich information. More explicitly: He shows that this socle is the direct sum of certain $D(\lambda^{(i)})$ and describes explicitly which $\lambda^{(i)}$ occur (II.0.5), and shows that at most p of them occur and that they belong to distinct blocks of Σ_{n-1} (III.3.1). Furthermore he determines the λ for which the restriction of $D(\lambda)$ to Σ_{n-1} is semisimple (II.0.6). These results include proofs (and in one case a correction) of conjectures that D. Benson had made for $p = 2$ [in *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, 381–394, Proc. Sympos. Pure Math., 47, Part 1, Amer. Math. Soc., Providence, RI, 1987; MR0933374 (89b:20037)]. As a special case one gets a criterion for the restriction of $D(\lambda)$ to Σ_{n-1} to be simple. Such a criterion had been conjectured by Benson for $p = 2$ [op. cit.] and by J. C. Jantzen and G. M. Seitz for arbitrary p [Proc. London Math. Soc. (3) **65** (1992), no. 3, 475–504; MR1182100 (93k:20026)]. It was proved first by Kleshchev in an earlier paper [Proc. London Math. Soc. (3) **69** (1994), no. 3, 515–540; MR1289862 (95i:20065a)]. Simpler proofs were then found by Ford [Bull. London Math. Soc. **27** (1995), no. 5, 453–459; MR1338688 (96g:20015)] and by Kleshchev (here in I).

The proofs of the results require a fair amount of combinatorics involving partitions, especially in III. The main technique, however, is to first prove theorems on representations of the special linear group SL_n and then to deduce results on the symmetric groups using the Schur functor: If G is a reductive algebraic group, then its Weyl group acts on the 0 weight space of any G-module. If one takes $G = \mathrm{GL}_n$, then one gets thus a functor from GL_n-modules to Σ_n-modules; this is the Schur functor (or a special case of it), cf. Chapter 6 of J. A. Green's *Polynomial representations of* GL_n [Lecture Notes in Math., 830, Springer, Berlin, 1980; MR0606556 (83j:20003)]. In I.2.12 Kleshchev establishes a dictionary that allows him to translate results from GL_n (actually SL_n, which works equally well) to Σ_n. He then has to investigate in great detail for each positive root α the $\mu - \alpha$ weight space in an SL_n-module of highest weight μ. In a recent manuscript ("On decomposition numbers and branching coefficients for symmetric and special linear groups") Kleshchev has determined bases for these weight spaces. For the purposes at hand in I–III somewhat less detailed results suffice.

Kleshchev's work on the tensor products with the sign representation settles an important long-outstanding problem. His work on restrictions from Σ_n to Σ_{n-1} is not only a useful tool for that problem, but is also of great interest in itself.

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