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Embeddings in manifolds, by Robert J. Daverman and Gerard A. Venema, Graduate Studies in Mathematics, Vol. 106, American Mathematical Society, Providence, RI, 2009, xviii+468 pp., ISBN 978-0-8218-3697-2, hardcover, US $75.00

The book *Embeddings in Manifolds* considers compacta, polyhedra, and manifolds and their embeddings into manifolds, with the hope of understanding just how complicated such embeddings can be. The target manifolds are usually assumed to be of high dimension, where there is sufficient room to maneuver. R. H. Bing explained the dimension situation in this way: "Dimension 4 is the most difficult dimension. It is too *old* to spank, the way we might deal with the little dimensions 1, 2, and 3; but it is also too *young* to reason with, the way we deal with the grown-up dimensions 5 and higher."

Just to be up front about things: A *manifold* is a space that is locally homeomorphic (= topologically equivalent) with Euclidean space, yet globally may be twisted in some interesting way like the Möbius band. An embedding $f : X \to Y$ of space X into space Y is a function that is a homeomorphism, not of X onto Y, but of X onto the image $f(X)$ of X in Y.

The most classical of embedding problems is answered by the Jordan Curve Theorem. The Jordan Curve Theorem states the fact (obvious yet nevertheless difficult to prove) that every simple closed curve (embedded circle) in the plane separates the plane into two pieces, namely an interior and an exterior, and it is the boundary of each. Algebraic topology was originally developed in the process of extending this theorem and others like it to high dimensions. Among the famous extensions are the Jordan Separation Theorem, the No Retraction Theorem, Brouwer's Fixed Point Theorem, and the Invariance of Domain Theorem.

The more difficult 2-dimensional Schoenflies Theorem, of which the Jordan Curve Theorem is an immediate corollary, states that all embeddings of the circle into the plane are equivalent from the standpoint of topology. That is, if $f : \mathbb{S}^1 \to \mathbb{R}^2$ is any embedding of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ into the plane $\mathbb{R}^2$, then the map f extends to a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ from the plane to itself. One of the many possible proofs of the Schoenflies Theorem comes from the theory of complex variables. The Riemann Mapping Theorem maps the interior of the curve onto the interior of the unit circle conformally, and Carathéodory [Car] showed that such a conformal mapping extends to a homeomorphism from the curve to the circle. The exterior of the curve is handled similarly. The Schoenflies Theorem fails for 2-dimensional spheres embedded in 3-dimensional Euclidean space $\mathbb{R}^3$ because the 2-sphere can be locally infinitely knotted in $\mathbb{R}^3$. Among the early examples of complicated embeddings were Antoine's Necklace [Ant], [AleJ2] and Alexander's Horned Sphere [AleJ1]. The goal of the book is to understand such difficulties in precise terms.

Antoine's Necklace [Ant] is a Cantor set in $\mathbb{R}^3$ whose complement is not simply connected. Such a Cantor set is called *wild*. The standard middle-thirds Cantor set is called *tame*. Antoine's Necklace is formed as the intersection of chains of chains

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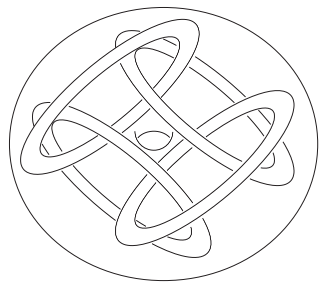

Figure 1. Antoine's Necklace

of chains (etc.) of linked solid tori, the first chain linked in the first solid torus as in Figure 1. Each smaller torus is likewise circled by a linked chain of smaller solid tori, each of which in turn is circled by a linked chain of even smaller solid tori, and so on to infinity. Alexander [AleJ2] noted that this necklace could be embedded in a 2-sphere in $\mathbb{R}^3$ by running feelers to the Cantor set.

Alexander's Horned Sphere [AleJ1] is a 2-sphere in $\mathbb{R}^3$ whose exterior is not simply connected. It is formed by running linked feelers to a limiting tame Cantor set, as in Figure 2.

The theorems in the book are basically of two types: The first shows that embeddings are more complicated (worse) than we could ever have imagined. The second type shows that embeddings are really nicer than we ever hoped, or can be approximated by nice embeddings, or at least with an additional hypothesis are nice. It is like the old shaggy dog story, where the dog owner is assured by one and all that the dog is really, really shaggy, though, at the end, the king assures the owner that the dog is really not so shaggy after all.

Some authors have asserted that topologists have taken a wrong turn with a definition of embedding that allows for such complicated things, and these authors have proposed various fixes.

The most common fix is to assume that the embedding is smooth (infinitely differentiable) or piecewise linear (that is, polyhedral). But even those hypotheses

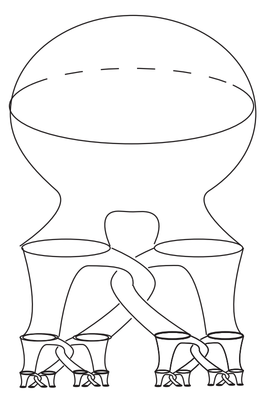

Figure 2. Alexander's Horned Sphere

allow some rather strange phenomena. For example, there is a 5-dimensional polyhedron P formed from a rather small number of polyhedral pieces (simplexes) that is topologically equivalent to the 5-dimensional sphere $\mathbb{S}^5$ and yet has the following properties: There are four simplex edges in P that form a simple closed curve J. As viewed in the polyhedron P, the edges of J are straight segments. But as viewed in the topologically equivalent 5-dimensional sphere $\mathbb{S}^5$, the curve J is infinitely knotted in the neighborhood of each point. The homeomorphism $P \to \mathbb{S}^5$ takes some straight arcs to infinitely knotted arcs. (See [Edw1] and [Can].)

Another suggested fix is to abandon the geometric picture entirely and turn instead to algebra where a typical algebraic realization might be a *supposedly well-behaved* direct or inverse limit. But such limits are precisely the algebraic analogues of infinite knotting.

It seems to us that a more reasonable point of view is to accept these infinitely knotted embeddings as a necessary consequence of the completion process that passes from the rational numbers to the real numbers. Complicated embeddings arise in the same limiting process that gives us those wonderful irrational numbers. The unification and simplification process of completion has complicated consequences. The consequences are to be loved, or at least appreciated or acknowledged. Paul Alexandroff [AleP] wrote, "I would formulate *the* basic problem of set-theoretic topology as follows: *To determine which set-theoretic structures have a connection with the intuitively given material of elementary polyhedral topology and hence deserve to be considered as geometrical figures—even if very general ones.*"

The problems considered in the book are the following:

Main Problem. Which embeddings of one topological space X in another topological space Y are equivalent? That is, given two embeddings $f_1, f_2 : X \to Y$, when is there a homeomorphism $h : Y \to Y$ such that $f_2 = h \circ f_1$?

Taming Problem. If $f : X \to Y$ is an embedding from a polyhedron X into a manifold Y, is f equivalent to a piecewise linear embedding? (An embedding that is equivalent to a piecewise linear embedding is called *tame*. An embedding that is not equivalent to a piecewise linear embedding is called *wild*.)

Unknotting Problem. Which piecewise linear embeddings of a polyhedron X into a manifold Y are equivalent? (The term *unknotting* refers to the common occurrence where there is a standard embedding considered to be unknotted, and the unknotting problem asks whether a given embedding is equivalent to this unknotted embedding.)

As noted by the authors, "While isolated examples of wild embeddings were discovered earlier, the work of R. H. Bing in the 1950s and 1960s revealed the pervasiveness of wildness in dimensions three and higher." The authors give examples of wildness in all high dimensions and all codimensions. (The *codimension* of an embedding $f : X \to Y$ is the integer $k = \dim(Y) - \dim(X)$. The codimension measures the extra room available for attempts to move the image of X around in Y.)

While the work of Bing initiated the considerable efforts to understand tame and wild embeddings, Bing's fundamental work in dimension three is mostly side stepped by the book. His work, and that of those who followed in dimension three, relied on very difficult and fundamental results that are outside the scope of this book. In particular, the Moise [Moi1], [Moi2] and Bing [Bin1] proofs of the triangulability of 3-manifolds and the accompanying *Hauptvermutung* for 3-manifolds, Bing's Side Approximation Theorem [Bin2], Bing's Taming by Homeomorphic Side Approximation [Bin3], and Bing's 1-ULC Taming Theorem in dimension 3 [Bin4], which are necessary in dimension three, do not appear in this book, though they and some of their consequences are occasionally mentioned. Some of their generalizations to higher dimensions do appear. But the omissions form a gap waiting to be filled by some ambitious writer.

Standard results from algebraic topology (homotopy theory, homology, and cohomology) and standard results from piecewise linear topology (simplicial complexes, general position, regular neighborhoods) form the basic prerequisites for the book. The additional techniques that are characteristic of the subject involve cut and paste arguments, careful inductions on dimension, and infinite applications of such processes, with the attendant difficult attention to convergence.

The key technical tool developed in the book is that of *engulfing* [Sta1]. Engulfing is a method designed to stretch an open set, using regular neighborhood theory, to surround a critical subcomplex. The simplest case simply stretches the open set along the product of a complex with an interval. Such a product arises from a homotopy that is placed in general position. Complications arise since the homotopy may not be embedded. Delicate inductive arguments allow considerably more complicated stretchings. The method is powerful enough that Stallings was able to use it to prove the Poincaré Conjecture in high dimensions [Sta2]. There are many other consequences.

After giving many examples of wild embeddings in Chapter 2 and developing engulfing and its consequences in Chapter 3, the authors proceed in succeeding chapters to prove the best-known taming and approximation theorems in high dimensions. The extent of the material to be covered is indicated by the fact that the authors include 15 pages of references. The subject divides naturally into cases according to *codimension*, where, as the reader will recall, the codimension of the embedding $f : X \to Y$ is the integer $k = \dim(Y) - \dim(X)$. Generally speaking, results are easier when k is large and things are most complicated when $k = 2$, the codimension in which classical knotting occurs, as with the embedding of the circle into 3-dimensional space. This classical knotting can always be iterated in the small, so that infinitely knotted objects arise because of codimension 2 classical knotting. In general, tame embeddings are characterized by the local triviality of various homotopy conditions. The authors have to rely on more and more difficult external results that do not appear in the book as codimensions shrink toward 0.

Of all the beautiful and complex methods that one encounters in the book, my particular favorite is the method of Štan'ko [Stan], a method that works in codimension 3. It is an extremely clever and beautiful example of the kind of infinite processes, with convergence issues, that arise in the subject. M. A. Štan'ko observed that, if we restrict our view to a 3-dimensional slice in a high-dimensional manifold, then the visible portion of any codimension 3 compactum seems to be knotted in a very simple way. To see this in the first place requires a good deal of imagination and insight. Štan'ko then notes that one can tear apart that knotting by ripping one portion of the compactum through another portion. As far as the 3-dimensional slice is concerned, this results in a new embedding of the visible portion of the compactum, and the new embedding has simpler knotting. Unfortunately, adjacent to the slice, one creates intersections. However, the intersections can be avoided by an adjacent, yet smaller, preconditioning move of exactly the same sort . . . , which creates new smaller intersections, which can be avoided by adjacent smaller preconditioning moves of the same sort, etc., etc., etc. The limit can be managed in such a way that the result is an unknotted embedding of the compactum.

The Štan'ko paper is four pages long. I heard about it via the grapevine in a path that eventually led back to Bob Edwards [Edw2] who had learned about it from Štan'ko's paper. It took me a month to digest the four pages. Edwards noted that the method can be reinterpreted in terms of an infinite 2-dimensional complex that models the algebraic object that is a commutator of commutators of commutators, infinitely. D. R. McMillan called such an algebraic object an *omegatator*. Daverman and Venema call the geometric realization a *Štan'ko complex*. It is more commonly known in the literature by the name suggested by McMillan: *the grope*. The grope plays an interesting role both in the approximation of codimension 1 spheres by tame spheres [AncCan] and in the proof of the Double Suspension Theorem [Edw1], [Can], which states that the double suspension of any homology sphere is a topological sphere.

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