BOOK REVIEWS

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Conformal fractals: ergodic theory methods, by Feliks Przytycki and Mariusz Urbański, London Mathematical Society Lecture Note Series, 371, Cambridge University Press, Cambridge, 2010, x+354 pp., ISBN 978-0-521-43800-1

1. The myriad roles of measures

An almost universal setting for dynamics involves a topological space X equipped with a measure μ. The domain of μ is referred to as the Borel σ-algebra of $X, \mathcal{B}$, which is the smallest collection of sets containing the open sets of X and closed under taking countable unions and complements. Typically, the setting suggests a method of measuring the open sets; however, the diversity of such measurements makes the subject of measure theory interesting and complex. In Euclidean space we use the volume of a box; on manifolds we use the local volume of a small ball as determined by a Riemannian metric. On a topological group, the Haar measure is the unique measure invariant under (left or right) group translation. On a metric space a measure often arises from the "easiest" way to assign volume to a ball. In short, a measure provides a notion of size generalized to badly misshapen sets.

Many abstract spaces have natural measures; in the symbol space $X = \prod_{j=0}^{\infty}\{0,1\}$, in order to define and measure a ball centered at $\omega \in X$, which we denote by $B_r(\omega)$, infinite strings of symbols are matched up against ω to find the smallest index where they disagree to get the radii and a measure for the ball. The points whose first k coordinates are $\omega_0, \omega_1, \ldots, \omega_{k-1}$ compose $B_{2^{-k}}(\omega)$, and since there are 2^k disjoint balls with this radius, a good choice is to set $\mu(B_{2^{-k}}(\omega)) = 2^{-k}$. The Carathéodory construction of a measure for a locally compact topological space X allows us to obtain a measure from a large variety of definitions of the "size of a basic open set". The resulting measure has as its domain the Borel sets, and we end up with a natural and elegant notion of measuring sets on X which goes well beyond our notions of volume of balls and cubes in Euclidean space. This of course allows us to integrate Borel measurable functions on X.

When X is compact, we usually normalize μ so that $\mu(X) = 1$ and call μ a probability measure. This brings us to the *raison d'être* of measures. They carry information well beyond volume. The probabilistic role of measures was noticed by Boltzman and Gibbs in their attempts to understand the macroscopic behavior of a closed system containing gas particles in the 1800s (see, e.g., [21]). The paradox was that the particles were acting under known forces, yet following 10^{27} of them seemed impossible. Moreover, instead of the classical invertible flows one might

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observe, such a system tended towards an equilibrium state, a uniform distribution of the gas throughout the container which was best represented by a measure.

A third important role of a measure is that it can "pick out" a dimension. These measures are often called geometric measures. Consider the interval $I = [a, b]$ as a subset of $\mathbb{R}^2$. While the 2-dimensional Lebesgue measure of I is clearly 0 since I can be covered by finitely many rectangles of arbitrarily small area, the 1-dimensional Lebesgue measure is its length. The usefulness of measures to pick out more subtle notions of dimension is most clearly evident with Hausdorff measure. It provides an effective tool for detecting a noninteger dimension; if we consider the ternary Cantor set C as a subset of I, then taking countable open covers and defining the outer measure $m_s^*(C) = \inf\{\sum_k (\text{length}(J_k))^s : C \subset \bigcup_k J_k,\ J_k \text{ an interval}\}$, it is not hard to calculate that for $s < \ln 2/\ln 3$, $m_s^*(C) = \infty$, while for any $s > 2/3, m_s^*(C) = 0$.

Measures play a central role in the theory of communications, or information theory as first outlined by Claude Shannon in 1948 [16]. If a message is imperfectly transmitted and the received message is "Meet me ths Fridy", one would easily know how to fill in the blanks. (FYI, this has been raised to a fine art form by today's texting and tweeting.) The redundancy is reflected by a measure associated to the English language in which letters are not equally likely to occur. Indeed there's a Markov measure that helps determine what letter might occur next given one that has just occurred; the classic examples trotted out being what occurs after a Q and, say, the limited number of letters one can expect after an X.

Another application occurs in Ramsey theory, by which we mean the study of extracting order from apparent chaos; in particular, one uses measures to pick out patterns from seemingly random blocks of numbers. Nowhere is this more apparent than in the work of Furstenberg [7, 8], where measure-preserving ergodic theory techniques are used to give a new proof of an important number-theoretic result, the Szemerédi Theorem. The large body of work was followed by many others, and more recently, by Tao in [18], where some of the process is reversed to offer new techniques for obtaining strong recurrence results on measure-preserving transformations under quite random assumptions. This is not a topic explicitly covered in the book under review, so we refer to the references mentioned in [18] for a full account.

More often than not the measure on a space X is being studied in conjunction with a dynamical system which is iterated infinitely often to model the passage of time. If $T : X \to X$ and μ is a Borel measure on X, one frequently requires that $\mu(T^{-1}A) = A$ for any $A \in \mathcal{B}$. Demanding the invariance of the measure is physically inspired, but if, say, X is a k-dimensional Riemannian manifold and μ is the k-dimensional volume form, then at the very least for a diffeomorphism T on X we have that the zero sets are preserved under T; $\mu(A) = 0$ if and only if $\mu(T^{-1}A) = 0$. A measure that preserves only the zero sets is called nonsingular, and a complete classification (up to orbit equivalence) of nonsingular invertible transformations in the 1970s and 1980s borrowed from and lent to the solution of an important classification problem of von Neumann factors (up to isomorphism) [3, 4, 11].

It is a classical result of Krylov and Bogolubov (see, e.g., [10, Thm. 1.4.3] that if T is a continuous map from a compact space X to itself, then each point $x_0 \in X$ gives rise to an invariant measure as follows. Let δ_{x_0} denote the point mass measure at x_0; i.e., given $A \in \mathcal{B}$, $\delta_{x_0}(A) = 1$ if $x_0 \in A$, and 0 otherwise. Then the sequence of point mass measures $\mu_n = \frac{1}{n}\sum_{k=1}^{n-1}\delta_{T^k x_0}$ always has weak-* limit points, each of which can easily be seen to be an invariant measure. However, a measure obtained this way does not always provide the best tool for studying the dynamical properties of T, for example if x_0 is periodic.

Invariant measures flow from many mathematical and physical sources. Statistical mechanics was introduced by the physicists Maxwell, Boltzman, and others to understand the time evolution of systems of fluids or gases. The abundance of particles leads one to assume the inefficiency of following each particle's motion individually, and the system seems to unfold over time almost randomly, achieving an equilibrium state independent of the beginning state. This idea has been developed into a mathematical theory of thermodynamical formalism, where physics terminology has been adapted to a purely mathematical setting, with a dictionary connecting the fields (see, e.g., [10] or [21]). Gibbs states correspond to invariant measures which maximize a distribution, potentials correspond to probability distributions, and pressure is a function on potentials which satisfies some useful variational principles.

At this point in the discussion of measures, all sorts of questions appear: What invariant measures are natural for a given system? When there are many invariant measures, what is the relevant information carried by each? When a measure ν on X is nonsingular with respect to T, how do we detect an invariant measure μ with the same measure 0 sets as ν, or show that none exists?

2. Julia sets of rational maps of the sphere

There is a great conflation of these ideas where different aspects of measure theory, analysis, and geometry interact beautifully, one informing the other, in the field of complex dynamics. For a rational map of the Riemann sphere $\mathbb{C}_\infty$, say $R(z) = \frac{p(z)}{q(z)}$, with no common factors and $d = \min\{\deg(p), \deg(q)\} \geq 2$, the Julia set is well known to be typically a fractal set supporting a variety of interesting measures. The Fatou and Julia sets are defined by $F(R) = \{z \in \mathbb{C}_\infty : \{R^n\}_{n\in\mathbb{N}}$ is equicontinuous at $z\}$, and $J(R) = \mathbb{C}_\infty \setminus F(R)$. In general, the surface area measure on the sphere (Lebesgue measure) is not the one of interest; rather there is a unique measure of maximal entropy (which is $\log(\deg(R))$) that gives equal weight to each preimage of a point ([6, 12, 13]), and for many maps R, there exists a unique nonatomic conformal measure, which picks out the dimension of the Julia set. Given $t \in \mathbb{R}$, a Borel probability measure ν is called a t-conformal measure if it is supported on $J(R)$ and it satisfies

$$\nu(R(B)) = \int_B |R'(z)|^t d\nu$$

for every Borel set B such that $R|_B$ is injective. In this setting we are interested in the value t corresponding to the Hausdorff dimension of $J(R)$. In a landmark result, it was shown that μ and ν are completely singular [20], except in rare circumstances which have interesting algebraic origins.

Simple smooth and fractal Julia set examples illustrate the ideas of these "natural measures" associated to a rational map. It is well known that every degree 2 rational map is conformally conjugate to one of the form $R_{(a,b)} = a(z+1/z+b)$ with $a, b \in \mathbb{C}$, $a \neq 0$. We look inside a sample family of the form $T_a = a(z+1/z-2)$. The critical points of any of these maps are $c = \pm 1$, so we can control certain dynamical and

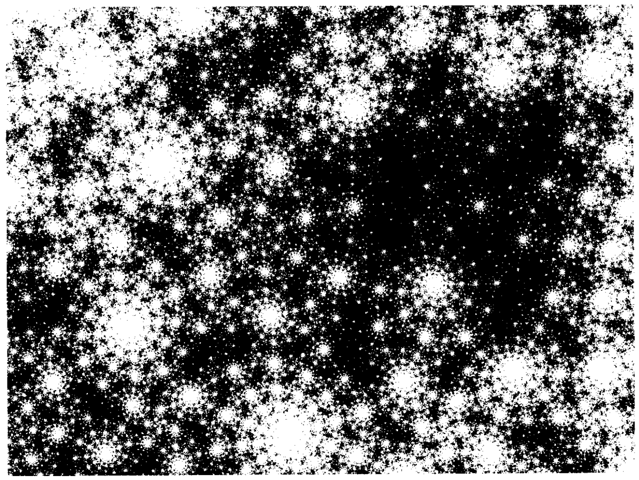

FIGURE 1. The distribution of the measure μ of maximal entropy for T

measure-theoretic behavior by our parameter choices. The map $T := T_{-\frac{1}{4}+\frac{i}{2}}$ maps the critical point $c_1 = 1$ to $c_2 = -1$ under two iterations, which is then mapped to the repelling point at ∞. It is well known that if all critical points terminate in repelling fixed points, the Fatou set is empty, so $J(T) = \mathbb{C}_\infty$. Now we consider two natural invariant measures for T with support $\mathbb{C}_\infty$. First, there is the unique invariant measure of maximal entropy, call it μ, enjoying the property that the measure-theoretic entropy $h_\mu(T) = \log 2$. Then it can be shown that the condition of finite forward critical orbits gives enough expansion so that there is also an invariant probability measure $\nu \sim m$, where m denotes normalized surface area.

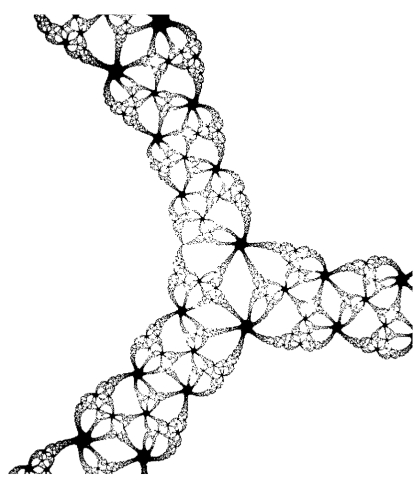

FIGURE 2. $J(T_a)$ is the black fractal set, and μ is supported on $J(R_a)$; $a = -.4 + .78i$.

In Figure 1 we see the distribution of the maximal entropy measure μ by using an algorithm that randomly chooses a preimage, with equal probability, and plots it [9]. Its support is the entire sphere, but its singularity with respect to Lebesgue measure is evident.

Using the value $a = -.4 + .78i$, we find an attracting period 3 cycle which necessarily attracts the critical point at 1; so in Figure 2 we see $J(T_a)$ in black, showing the support of the various available natural measures. Mutually singular invariant measures highlight different dynamical and statistical properties of orbits of points, because the points are selected "randomly" from disjoint sets of full measure.

3. The book under review

The book is an ambitious effort to apply the many roles of measures to a fairly wide variety of settings of expansive transformations. It covers a lot of ground, much of which is hinted at in the short first chapter entitled "Basic examples and definitions"; these run the gamut from symbolic dynamics, complex dynamics, Kleinian groups, symbolic shifts, and adding machines to Smale horseshoes. The early chapters provide either a terse introduction to, or a nice review of, the most important concepts in ergodic theory and dynamical systems, depending on whether or not the reader has some prior familiarity with the field. The basic examples and definitions are all present; for example, Chapter 2 provides a concise synopsis of ergodic theory, including in its coverage noninvertible and non-measure-preserving results, important topics for exploring examples that occur naturally in mathematics. The book takes a foray into one real dimensional dynamics, giving a thorough account of embeddings of the Cantor set into the real line and applying the treatment to the Feigenbaum Universality phenomenon; in particular, proofs of results on infinitely renormalizable C^2 interval maps are shown. The chapter stops short of the complete picture involving the universal constant, but references for further study are given. The large number of contributions by both Przytycki and Urbański to the topics covered in the book are clear from, for example, the references in [5] and [19]. Much of the exposition consists of original proofs by the authors. Some cover topics not easily found in the literature (e.g., material from [15] and [17]). There are readable books on complex dynamics, such as Beardon [1], Carleson and Gamelin [2], or Milnor [14]; familiarity with the material in one of these books makes the general setting of this text easier to digest.

Conformal Fractals is packed with classical gems with proofs provided. The authors are experts on extending the subject in many of the important directions it has taken in the past several decades, especially the move from uniformly hyperbolic maps to expansive maps, which includes many rational maps of the sphere. This is an interesting text that could be used for a year-long graduate course in ergodic theory; the first three or four chapters contain enough information for a thorough course in ergodic theory and topological dynamics, while the remaining chapters contain a wide array of topics from which one could choose to develop an additional semester-long course. It is very densely written; brief accounts are given of topics that could fill many more chapters, but references and historical remarks send the reader out for further information. It is useful to have the breadth of the subject of fractal ergodic theory, including many topics beyond rational maps, in one volume. All the applications of measures that appear in the first section of this review are covered in great detail in the book, except for the Ramsey theory. However, recent activity indicates that this book may provide a useful tool in that area too, as multiscaling in fractals is akin to finding order from chaos. The book can also be used as an essential reference for the growing field of fractal ergodic theory; it provides a worthwhile addition to every ergodic theorist's library.

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