

SELECTED MATHEMATICAL REVIEWS

related to the papers in the previous section by
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MR0095844 (20#2342) 14.00

Whitney, Hassler

Elementary structure of real algebraic varieties.

Annals of Mathematics. Second Series **66** (1957), 545–556.

Definitions: In real n -space R^n , or in complex n -space C^n , the differential $df(p)$ of a function f at a point p is a covector, defined in the usual way. The rank of a set S of functions at a point p is the maximum number of independent differentials $df_1(p), \dots, df_s(p)$, the f 's being in S . The rank of a point set Q at $p \in Q$ is the rank at p of the ideal formed by all the polynomials (with real or complex coefficients) which vanish in Q . The set of common zeros (in R^n or C^n) of a set of polynomials is a (real or complex) algebraic variety V . An algebraic partial manifold M is a point set associated with an integer ρ , such that for any $p \in M$ there exists a set of polynomials f_1, \dots, f_ρ , of rank ρ at p , and a neighbourhood U of p , such that $M \cap U$ is the set of zeros of the f 's in U ; and $n - \rho$ is then the dimension of M .

For any V , let M_1 be the set of points $p \in V$ where the rank of V is its maximum. Then, both in the real and in the complex cases, it is proved that M_1 is an algebraic partial manifold, and that $V_1 = V - M_1$ is void or is a proper algebraic subvariety of V . The “splitting process” leading from V to $M_1 \cup V_1$ can now be applied to V_1 , if V_1 is not void, and comes to an end after a finite number of steps, giving

$$V = M_1 \cup M_2 \cup \dots \cup M_s$$

(where $s \leq 2^{n-1}$); here the M 's are algebraic partial manifolds two by two disjoint, and each point set $V_i = M_{i+1} \cup \dots \cup M_s$ is an algebraic variety, and so is closed. The M 's need not be closed (or connected), since each M_i may have limit points in later M 's. A different splitting (with at most n terms) can be obtained by manifolds of decreasing dimension.

It is further shown that, in the real case, each M_i in the previous splitting has but a finite number of topological components. More generally it is proved that, if V' is a subvariety (possibly void) of a real variety V , then $V - V'$ has at most a finite number of topological components; this is obtained by studying the connection between V and the smallest complex variety containing V .

From MathSciNet, June 2012

B. Segre

MR0148079 (26 #5588) 57.20; 57.50

Thom, René

La stabilité topologique des applications polynomiales.

L'Enseignement Mathématique. Revue Internationale. IIe Serie **8** (1962), 24–33.

If E and E' are two Euclidean spaces, two maps $f, g: E \rightarrow E'$ are said to have the same topological type if there are homeomorphisms h and h' of E and E' with themselves such that $h'f = gh$. The purpose of this note is to give an example of a

family of polynomial maps, depending on a parameter k , such that the topological types of the maps change continuously with k . Hence, the set of topological types (at least for dimensions greater than 3) has the power of the continuum.

The description of the example is given in detail. However, it depends on a deep theory of manifold collections and stratified maps. Such a theory, including more than is actually needed for the example, is sketched in this paper, but it seems that a detailed description of these interesting results will require a paper of greater length.

From MathSciNet, June 2012

D. W. Kahn

MR0188486 (32 #5924) 32.44

Whitney, Hassler

Local properties of analytic varieties.

Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), 205–244, Princeton Univ. Press, Princeton, N. J., 1965.

The author proposes to use various tangent cones to (complex) analytic varieties in the study of the local behavior of such varieties, and, in particular, in attacking the stratification problem which he discusses in the second half of the paper.

The most important of the six types of tangent cones at a point p on a variety V are the following: (a) v belongs to $C_3(V, p) = C(V, p)$ if and only if there is a sequence of points $\{q_i\}$ of V converging to p and a sequence of complex numbers $\{a_i\}$ such that $a_i(q_i - p) \rightarrow v$; (b) v belongs to $C_6(V, p)$ if and only if, for every germ of complex analytic functions f in the ideal $I(V, p)$ of germs of holomorphic functions at p vanishing on V , one has $df(p)v = 0$. The latter tangent cone (space) was introduced by H. Rossi [Ann. of Math. (2) **78** (1963), 455–467; MR0162973 (29 #277)]. All tangent cones are algebraic varieties, and differentials of holomorphic mappings of varieties map each tangent cone into the tangent cone of the same type (this solves, in particular, a problem posed by the reviewer [Proc. Conf. Complex Analysis (Minneapolis, 1964), Appendix, Problem 6, p. 305, Springer, Berlin, 1965; see MR0174781 (30 #4974)]). $C_6(V, p)$ can be identified with the space of derivations on the ring $O(V, p)$ of germs of holomorphic functions on V at p . Also, $C(V, p)$ can be characterized intrinsically using the interesting result that $C(V, p)$ is the set of common zeros of the initial polynomials of functions in $I(V, p)$.

For the study of stratifications, a so-called tangent space $\tau(V)$ to a variety V of constant dimension r in C^n is introduced. $\tau(V)_p$ is the set of all r -dimensional linear subspaces of $C_6(V, p)$ (considered as points in the Grassmann space $G^{n-1, r-1}$) which are limiting positions of tangent spaces to simple points. $\tau(V)$ is a variety in $C^n \times G^{n-1, r-1}$ whose fibers are algebraic (equal to a point at simple points of V).

A stratification of the variety V is an expression of V as the disjoint union of a locally finite set of connected analytic manifolds, called strata, such that the frontier of each stratum is the union of a set of lower-dimensional strata. It is shown that stratifications satisfying some additional regularity properties with respect to the tangent spaces $\tau(M_i)$ of the strata M_i exist. Now let p be a point in a stratum M and N the analytic plane orthogonal to M at p . A neighborhood U of p in C^n is fibered by a continuous function $\varphi: (U \cap M) \times (U \cap N) \rightarrow U$ if (a) $\varphi(q, p) = q$ for $q \in U \cap M$, (b) $\varphi(p, q) = q$ for $q \in U \cap N$, (c) U is the disjoint union of the fibers

$F(q) = \{q' = \varphi(p', q) : p' \in U \cap M\}$, (d) each $F(q)$ is biholomorphically equivalent to $U \cap M$, and (e) if $F(q)$ intersects a stratum, then it lies in that stratum. Such a fibration is semi-analytic if φ is analytic in the first variable, and it is analytic if φ is analytic. It is shown by examples (hypersurfaces which are nonhomogeneous along their set of singularities) that analytic fibrations may not exist for any stratification (thus refuting a conjecture of the reviewer [loc. cit., Appendix, Problem 7, p. 306]). It is conjectured that semi-analytic fibrations do exist for suitable stratifications. Using an interpolation procedure, the conjecture is proved for points in $(n - 2)$ -dimensional strata of hypersurfaces in C^n .

Among the many enlightening examples are varieties which are locally non-algebraic at every singular point.

From MathSciNet, June 2012

L. Bungart

MR0192520 (33 #745) 32.44; 32.47

Whitney, Hassler

Tangents to an analytic variety.

Annals of Mathematics. Second Series **81** (1965), 496–549.

Étude détaillée des propriétés différentiables (locales) des espaces analytiques complexes (réduits). Précisons la terminologie: les mots “manifold”, “submanifold” se traduisent en français par “variété”, “sous-variété”, tandis que “variety”, “sub-variety” se traduisent par “espace analytique”, “sous-espace analytique”. L'auteur considère des variétés, respectivement des espaces analytiques plongés dans C^n (i.e., fermés dans un ouvert $H \subset C^n$). Son étude complète un autre travail [*Differential and combinatorial topology*, pp. 205–244, Princeton Univ. Press, Princeton, N.J., 1965; MR0188486 (32 #5924)] mais elle se suffit à elle-même. Elle est divisée en trois parties: la partie I se borne à un rappel de résultats plus ou moins classiques. La partie II étudie les vecteurs tangents à un sous-espace analytique $V \subset C^n$, et notamment le cône tangent $C(V, p)$ en un point $p \in V$; $C(V, p)$ est formé des $V \in C^n$ tels qu'il existe une suite de $p_i \in V$ et de $a_i \in \mathbf{C}$ avec $\lim a_i(p_i - p) = v$; c'est un cône algébrique, défini par l'annulation des termes de plus bas degré du développement en série de polynômes homogènes des fonctions de l'idéal définissant le germe de V au point p (pris comme origine). En un point “simple” $p \in V$, $C(V, p)$ n'est autre que l'espace tangent en p . Le cône $C(V, p)$ est un foncteur covariant vis-à-vis des applications holomorphes de l'espace analytique V ; il est donc attaché intrinsèquement (à un isomorphisme linéaire près) au germe d'espace analytique abstrait. Si V est irréductible en p , $C(V, p)$ est connexe, mais non nécessairement irréductible. L'ensemble des couples (p, v) tels que $p \in V$ et $v \in C(V, p)$ n'est pas nécessairement un sous-espace analytique de $H \times C^n$, même si V est fermé dans l'ouvert $H \subset C^n$; mais il est contenu dans l'ensemble analytique $C_4(V)$, adhérence dans $H \times C^n$ de l'ensemble des (q, v) , où $q \in V_{sp}$ (ensemble des points simples de V supposé de dimension constante), et v est tangent à V en q . Dans les trois derniers paragraphes de la partie II, l'auteur étudie les propriétés d'un espace analytique V au voisinage d'une sous-variété (“manifold”) $M \subset V$: il lui associe un espace analytique $K_M(V)$ de même dimension que V , fibré sur M , dont les fibres sont des cônes algébriques; l'ensemble des $p \in M$ tels que la dimension de la fibre au-dessus de p soit au plus égale à un entier donné est un sous-espace analytique

de M . En supposant $r = \dim V > \dim M = m$, l'auteur associe une sous-variété différentiable-réelle B , de dimension réelle $2m + 1$, contenant M et contenue dans V , et appelée une aile (“wing”) de M dans V . Étant donné un sous-espace analytique $V' \subset V$ ($\dim V' \subset \dim V$), on peut trouver une “aile” d’un ouvert non vide $M' \subset M$, arbitrairement voisin d’un $p \in M$ donné, telle qu’elle ne rencontre pas V' . Le cas où $r = m + 1$ est spécialement étudié: il existe alors un fermé $Q \subset M$ sans point intérieur, tel que, au voisinage de chaque point de $M - Q$, V “s’attache différentiablement à M ”.

La partie III étudie d'une part les r -plans tangents à l'espace analytique V de dimension r , d'autre part les “stratifications” d'un espace analytique. L'adhérence, dans $V \times G$, de l'ensemble des couples $(q, T^*(V, q))$, où q est un point simple de V , et $T^*(V, q)$ l'élément de la grassmannienne G défini par le plan tangent à V en q , est un espace analytique $\tau^*(V)$. La démonstration utilise la cohérence du faisceau d'idéaux de V . Les fibres de l'application $\tau^*(V) \rightarrow V$ sont des sous-ensembles algébriques de G ; l'auteur ne compare pas ces fibres aux cônes tangents.

Une stratification d'un espace analytique V est une partition de V en sous-ensembles qui sont des variétés (“manifolds”) possédant la propriété suivante: pour toute variété M la partition, l'adhérence \overline{M} est un sous-espace analytique dont M est l'ensemble des points simples de dimension maximum, et \overline{M} est réunion des variétés M' de la partition qui rencontrent \overline{M} . Un ensemble analytique V possède une stratification naturelle, définie par récurrence sur $\dim V$: V est réunion de la variété M des points simples de V de dimension maximum, et de $V - M$ qui est un ensemble analytique de dimension strictement plus petite que $\dim V$, ensemble analytique dont on prend la stratification naturelle. Une autre stratification naturelle s'obtient en décomposant chaque variété de la stratification précédente en ses composantes connexes; ces deux stratifications de V sont localement finies. On démontre, en utilisant un classique théorème de Remmert-Stein: pour qu'une partition localement finie d'un espace analytique V , formée de variétés, soit une stratification, il suffit qu'elle possède la “propriété de frontière” (i.e., si M et M' sont deux ensembles distincts de la partition tels que M' rencontre \overline{M} , alors $M' \subset \overline{M}$ et $\dim M' \subset \dim M$). L'auteur introduit la notion (assez fine) de “stratification régulière”; la stratification naturelle de V n'est pas toujours régulière, mais toute stratification possède une stratification régulière plus fine (théorème 19.2).

Tels sont quelques-uns des principaux résultats de cette étude qui en contient beaucoup d'autres.

From MathSciNet, June 2012

H. Cartan

MR0239613 (39 #970) 57.20; 55.00

Thom, R.

Ensembles et morphismes stratifiés.

Bulletin of the American Mathematical Society **75** (1969), 240–284.

The author develops the category of “stratified” sets and morphisms in order to alleviate some of the disadvantages of the category of simplicial complexes and maps.

Let X, Y be two paracompact smooth manifolds and let $f: X \rightarrow Y$ be a submersion (a differentiable map, everywhere of maximum rank equal to $\dim Y$). A

submersion f is said to be controlled if there exists a control function $g: X \rightarrow R$, that is, a positive real C^∞ function such that (1) for some $a > 0$ the differential dg is not zero at any point of $g^{-1}(]0, a])$ and (2) the restriction of f to the manifold-with-boundary $g \geq e$ is a proper submersion into y for each positive $e < a$.

If f is controlled by g , then the stratified space $M(f)$ generated by f is defined to be the disjoint union $X \cup Y$ with the topology having as a subbase the open sets of X and all sets of the form $[(g < e) \cap f^{-1}(V)] \cup V$ as V ranges over the open sets of Y .

A stratified set E is given as a finite union of connected C^∞ manifolds, the strata X_i of E . It is so defined as to be a stratified space and can be obtained from the disjoint union of the X_i by a system of equivalence relations that “glue” the strata together. The defining properties of the equivalence relations lead to the concept of incidence relations among the strata, analogous to the incidence relations in a simplicial complex.

The theory of stratified morphisms $E \rightarrow E'$ is developed and discussed primarily for compact stratified sets.

The general coverage of the paper is suggested by the main headings: (I) Topological theory; (II) Euclidean theory; (III) Analytic sets; (IV) Theory of singularities of differentiable maps. In IV, a brief indication is given of how the theory of stratified sets and morphisms can be applied to the local theory of differentiable maps.

From MathSciNet, June 2012

S. S. Cairns

MR0455018 (56 #13259) 58C25; 57D45

Mather, John N.

How to stratify mappings and jet spaces.

Singularités d'applications différentiables (Sem., Plans-sur-Bex, 1975), 128–176.
Lecture Notes in Mathematics, Vol. 535, Springer, Berlin, 1876.

This paper is, in a certain sense, a continuation of an earlier paper [the author, *Dynamical systems* (Proc. Sympos., Univ. of Bahia, Salvador, 1971), pp. 195–232, Academic Press, New York, 1973; MR0368064 (51 #4306)]. In this earlier paper, the author outlined a proof of the topological stability theorem, which states that topologically stable mappings are dense among smooth mappings when the source manifold is compact. The outline indicated the principal ideas and techniques involved but, in general, omitted the details. The central goal of the present paper is to provide detailed proofs of three key results stated in the earlier paper, including the topological stability theorem. We describe these main results but refer the reader to the review of the above paper for further details of the theory.

A fundamental idea of the theory is that a large class of mappings can be stratified. In fact it is proved that there is a natural stratification for a large class of smooth mappings which includes the proper infinitesimally stable ones. The stratification is first constructed for certain polynomial mappings and its minimality properties imply that it gives a globally well-defined stratification.

For the topological stability theorem, mappings of finite singularity type are considered. These are mappings which have global unfoldings to proper infinitesimally stable mappings. They form a dense set, and the dense subset of these mappings

which are transverse to the stratification of their stable unfoldings are proved to be topologically stable.

Lastly, a more refined local stratification procedure is applied to obtain a canonical stratification of the jet space. Mappings multi-transverse to this stratification are also proved to be topologically stable.

{The proofs of (2) and (3) are “sketched” in places and the reader should not only be familiar with the above-cited paper but also with the author’s series of papers on C^∞ -stability of mappings. Also, in § 9, part (a) of Proposition 9.1, Proposition 9.2 and the definition of general position contain errors.}

{For the entire collection see MR0438373 (55 #11287).}

From MathSciNet, June 2012

James Damon