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The stability of matter in quantum mechanics, by Elliott H. Lieb and Robert Seiringer, Cambridge University Press, Cambridge, 2010, xv+293 pp., ISBN 978-0-521-19118-0

Stability of matter is a fundamental fact about the nature of ordinary matter. In essence it says that macroscopic objects exist! It is at the same time a rigorous mathematical statement in the theory of quantum mechanics. I will describe its precise meaning below. It is somewhat surprising that stability of matter is not a subject treated in standard physics textbooks. It is however one of the most celebrated results and a cornerstone in mathematical physics. The book under review, *The Stability of Matter in Quantum Mechanics* by Lieb and Seiringer, is the first to give a complete and thorough account of stability of matter. I will begin with an overview of the subject itself.

The reason stability of matter is not treated in physics textbooks is not because of its lack of importance, in fact, what could be more important? More likely, the reason is that it is not easy to derive. In contrast to most other results in mathematical physics there was, to the best of my knowledge, no heuristic derivation of stability of matter prior to the rigorous proof of the theorem, which appeared in 1967 in the seminal work of Dyson and Lenard [4]. Even Onsager's paper [13], which is probably the very first to address the issue, was mathematically correct and presented ideas used in many later works.

Stability is an important concept in physics and the notion is used in many contexts. One of the triumphs of the theory of quantum mechanics is that it explains the stability of atoms. The puzzling question settled by quantum mechanics is why the electrons in the atom do not simply collapse on top of the atomic nucleus due to their mutual electrical attraction. There are two ways to formulate this problem. We might ask why there is dynamic stability, i.e., why the motion is well defined for all times independently of the initial condition. Or we might alternatively ask why there is energetic stability, i.e., why the total energy cannot be arbitrarily negative, which it would be if the electrons were arbitrarily close to the nucleus. This is indeed what may happen in classical mechanics, where we have neither energetic nor dynamic stability.

It turns out that in quantum mechanics energetic stability implies dynamic stability. In technical terms if the energy is bounded below there is a natural realization (the Friedrichs extension) of the energy operator, as a self-adjoint operator, i.e., the Hamiltonian, on a Hilbert space. This operator generates the dynamics. Hence a study of stability in quantum mechanics may focus on energetic stability. This is the topic of the book by Lieb and Seiringer.

The energetic stability of atoms, or more precisely of the hydrogen atom (an atom with one electron) is usually explained, at least heuristically, in the first few pages of most textbooks in quantum mechanics. The explanation is based on *the uncertainty principle*. Likewise, stability of the hydrogen atom is proved in the introductory chapter of the book by Lieb and Seiringer. It is pointed out here, however, that,

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contrary to what is stated in most physics texts, the famous Heisenberg formulation of the uncertainty principle is, in fact, not very useful in order to conclude stability. For this purpose the *Sobolev inequality* is a better formulation of the uncertainty principle and is used in the book by Lieb and Seiringer to prove stability of the hydrogen atom.

Energetic stability, i.e., the fact that there is a lower bound to the energy, is referred to in the book as *stability of the first kind*. Stability of matter, also called *stability of the second kind*, is a more complicated notion relating to the energy of macroscopic systems. Individual atoms or molecules are relatively small systems with a few degrees of freedom. Macroscopic matter, however consists of an enormous amount of atoms, i.e., it is made out of a macroscopic number of nuclei and electrons. As an example one gram of hydrogen consists of approximately $6 \cdot 10^{23}$ (Avogadros' number) hydrogen atoms. Stability of the first kind only states that the energy of such a system is not arbitrarily negative. It does not address the issue of how negative it may be depending on the size of the system, e.g., measured by the number of particles. For macroscopic systems, however, it is important that the dependence of the energy on the size of the system is at most linear. The energy of twice an amount of a substance should be essentially twice the energy of the amount itself. This is *stability of matter*. It is closely related to the *extensivity of matter*, i.e., that the volume of a substance grows proportional to its quantity, otherwise a macroscopic number of particles would not take up a macroscopic volume. As obvious as this may sound, it is difficult to prove.

Contrary to stability of the first kind, stability of the second kind does not follow from the uncertainty principle alone. It requires also the *Pauli-exclusion principle*, i.e., the fact, to be explained below, that electrons are fermions and thus cannot occupy the same one-particle states. Without the exclusion principle, stability of matter fails. In fact, as first noted by Dyson [3], the energy of such a system would have a super-linear behavior as a function of particle number and the volume would, indeed, *decrease*; *more* particles would take up *less* space.

It was mentioned above that stability of matter is usually not treated in physics textbooks. There is however another case of stability due to the Pauli-exclusion principle which is known to any physicist. This is Chandrasekhar's famous theory [2] (for which he got the Nobel prize in 1983) of gravitational stability and instability of stars in their late evolutionary state as white dwarfs. Chandrasekhar's theory was given a rigorous formulation in [10, 11], and this is also covered in Lieb and Seiringer's book.

Besides being a problem of basic physical importance, the study of stability of matter leads to a wealth of beautiful mathematics. Topics such as variational calculus, potential theory, operator theory, spectral theory, Sobolev inequalities, and phase space analysis need to be brought together in order to arrive at a proof of stability of matter.

Let me briefly review the precise formulation of stability of matter and, as a guide to the reader of the book, indicate the main steps in its proof.

Matter is described as consisting of electrons and nuclei. All the electrons are identical with the same mass m and negative charge $-e$. The nuclei may be different and have different masses and (positive) charges. The charge of a nucleus is Ze, where the integer Z is the atomic number of the nucleus. The smallest nucleus is the hydrogen nucleus (a single proton) with $Z = 1$ and all naturally existing nuclei have $Z \leq 92$ corresponding to the elements in the periodic table.

Imagine that we have N electrons and M nuclei with atomic numbers $\underline{Z} = (Z_1, \ldots, Z_M)$. Let $E_{N,M}(\underline{Z})$ be the smallest possible (actually the infimum) energy of such a system. It depends on the nuclear charges, their masses, the mass and charge of the electron, and Planck's constant $\hbar$ (this is really Planck's constant divided by 2π). Stability of the first kind is the claim that $E_{N,M}(\underline{Z})$ is finite (not negative infinity). Stability of matter states that

$$E_{N,M}(\underline{Z}) \geq -\Xi(Z)(N+M), \tag{1}$$

where the constant $\Xi(Z)$ depends only on $Z = \max\{Z_1, \ldots, Z_M\}$, Planck's constant, and the mass and charge of the electron, but *not* on the masses of the nuclei. Establishing stability of matter with a constant independent of the masses of the nuclei is physically important. Nuclei are much heavier than electrons, and the energy per particle should not diverge as the masses tend to infinity. In other words we might as well think of the worst case scenario when the masses of the nuclei are all infinite. This is the case referred to as static nuclei.

To give the precise definition of $E_{N,M}(\underline{Z})$ we introduce the 3-dimensional coordinates of the electron positions $\underline{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{3N}$, and the nuclear positions $\underline{R} = (\mathbf{R}_1, \ldots, \mathbf{R}_M) \in \mathbb{R}^{3M}$. The state of the electrons is described by a complex valued wave function $\psi(\underline{X}, \underline{\sigma})$, where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$ denotes the internal spin degrees of freedom. Each σ_i can take q values. For physical electrons the spin is $1/2$ corresponding to $q = 2$, but in the discussion here q could be any positive integer. The wave function should be normalized, i.e., $\sum_{\underline{\sigma}} \int_{\mathbb{R}^{3N}} |\psi(\underline{X}, \underline{\sigma})|^2 d\underline{X} = 1$, where $d\underline{X} = d\mathbf{x}_1 \cdots d\mathbf{x}_N$.

The important Pauli-exclusion principle can now be formulated as the requirement that the wavefunction is *fermionic*, which means that it is antisymmetric under the interchange of $(\mathbf{x}_i, \sigma_i)$ and $(\mathbf{x}_j, \sigma_j)$ for any $i \neq j$.

The energy consists of two parts a *kinetic energy*, which is

$$T_\psi = \frac{\hbar^2}{2m} \sum_{i=1}^{N} \sum_{\underline{\sigma}} \int_{\mathbb{R}^{3N}} |\nabla_{\mathbf{x}_i} \psi(\underline{X}, \underline{\sigma})|^2 d\underline{X},$$

and a *potential energy*, which is

$$V_\psi(\underline{R}) = \sum_{\underline{\sigma}} \int_{\mathbb{R}^{3N}} V_{\mathrm{C}}(\underline{X}, \underline{R}) |\psi(\underline{X}, \underline{\sigma})|^2 d\underline{X},$$

where we have introduced the electrostatic Coulomb potential

$$V_{\mathrm{C}}(\underline{X}, \underline{R}) = -\sum_{i=1}^{N} \sum_{j=1}^{M} \frac{Z}{|\mathbf{x}_i - \mathbf{R}_j|} + \sum_{1 \leq i < j \leq N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} + \sum_{1 \leq i < j \leq M} \frac{Z^2}{|\mathbf{R}_i - \mathbf{R}_j|}.$$

For simplicity we here consider the case where all nuclear charges are equal, i.e., equal to the maximal value Z. There is a monotonicity argument showing that this is, indeed, the worst case. Finiteness of the kinetic energy T_ψ implies that ψ belongs to the Sobolev space $H^1(\mathbb{R}^{3N})$. For such a function all terms in V_ψ are also finite. The precise definition of the energy is then

$$E_{N,M}(\underline{Z}) = \inf \left\{ T_\psi + e^2 V_\psi(\underline{R}) : \ \underline{R} \in \mathbb{R}^{3M}, \psi \in H^1 \text{ fermionic, normalized} \right\}.$$

Note that the static nuclei are described only through their positions $\underline{R}$, which are optimized in order to minimize the energy.

Stability of matter (1) can be derived from inequalities on the kinetic energy T_ψ and the Coulomb potential V_{C}. The first fundamental inequality is the Lieb–Thirring [8] kinetic energy estimate for normalized fermionic wavefunctions ψ with q spin states

$$T_\psi \geq \frac{\hbar^2}{2m}\frac{K}{q^{2/3}}\int_{\mathbb{R}^3} \rho_\psi(\mathbf{x})^{5/3} d\mathbf{x}.$$

Here we have introduced the electronic density

$$\rho_\psi(\mathbf{x_1}) = N \sum_{\underline{\sigma}} \int_{\mathbb{R}^{3(N-1)}} |\psi(\underline{X}, \underline{\sigma})|^2 d\mathbf{x}_2 \cdots d\mathbf{x}_N.$$

Note that the normalization condition on ψ implies that $\int_{\mathbb{R}^3} \rho_\psi(\mathbf{x}) d\mathbf{x} = N$. A celebrated (still unsolved) conjecture of Lieb and Thirring [9] is that the best constant K in the above inequality is obtained from the semiclassical expression

$$K_{\mathrm{CL}} = \frac{(2\pi)^{-3}\int_{\{\mathbf{p}\in\mathbb{R}^3\,:\,|\mathbf{p}|\leq 1\}} \mathbf{p}^2 d\mathbf{p}}{\left((2\pi)^{-3}\int_{\{\mathbf{p}\in\mathbb{R}^3\,:\,|\mathbf{p}|\leq 1\}} 1 d\mathbf{p}\right)^{5/3}} = \frac{3}{5}(6\pi^2)^{2/3}.$$

The second ingredient in deriving stability of matter is to control the Coulomb potential energy. There are several approaches to this part. The simplest, which however does not lead to the best-known constant in (1), uses an estimate by Baxter [1] on the Coulomb potential. It states that for all $\underline{X} \in \mathbb{R}^{3N}$ and $\underline{R} \in \mathbb{R}^{3M}$ we have

$$V_{\mathrm{C}}(\underline{X}, \underline{R}) \geq -(2Z+1)\sum_{i=1}^{N} W_{\underline{R}}(\mathbf{x}_i), \qquad W_{\underline{R}}(\mathbf{x}) = \max_{j=1,\dots,M}\{|\mathbf{x}-\mathbf{R}_j|^{-1}\}.$$

Baxter proved this with probabilistic methods, but it can be derived [12] using potential theory and elaborating on the original ideas of Onsager [13]. Lieb and Seiringer give several stronger versions of this type of electrostatic inequality. For the discussion here this original version will suffice. The importance of the inequality is that the Coulomb potential which contains terms depending on pairs of electron coordinates is estimated by a sum of terms containing only individual electron coordinates. This leads to an estimate on the energy that can be expressed entirely from the electron density

$$T_\psi + e^2 V_\psi(\underline{R}) \geq \frac{\hbar^2}{2m}\frac{K}{q^{2/3}}\int \rho_\psi(\mathbf{x})^{5/3} d\mathbf{x} - e^2(2Z+1)\int \rho_\psi(\mathbf{x}) W_{\underline{R}}(\mathbf{x}) d\mathbf{x},$$

from which it is an easy exercise to derive (1).

It turns out that (1) holds also with $N+M$ replaced by M, i.e., only the number of nuclei. The reason is that N much larger than MZ would mean that the system is very far from being electrically neutral; in fact, it would be very negatively charged, and this is not energetically favorable. Such an argument sounds intuitively simple but is, in fact, rather subtle and has been a very active research area in mathematical physics and is still not fully understood. It is often referred to as the *ionization problem* because it may be rephrased as the question, what is the maximal negative ionization of a system? Because of its implications to stability of matter, Lieb and Seiringer use the opportunity to review what is known about this intriguing problem and, in particular, they prove the stronger version of (1).

We have briefly reviewed some of the basic ideas presented in great details and with beautiful clarity in essentially the first half of *The Stability of Matter in Quantum Mechanics*.

On a historical note the book does not contain the original proof of stability of matter by Dyson and Lenard [4]. It is closer in spirit to the later and more elegant approach of Lieb and Thirring [8]. This latter derivation was, however, based on Thomas–Fermi theory, which the book chooses to circumvent.

The stability of matter discussed up to this point is for nonrelativistic quantum mechanics. Relativistic effects and in particular the interaction with the electromagnetic field are important phenomena. The emission and absorption of light are processes of basic importance to the structure of atoms and it has been ignored in the discussion so far. Unfortunately, there is no complete mathematical theory describing relativistic quantum mechanics and the interaction of light and matter. Results on stability are known in several approximate models, and these are also described in detail in the book. Although these models do not claim to be complete, they contain the basic feature, believed to be correct for all relativistic models, that *instability* occurs in certain ranges of the physical parameters. Extensions of stability of matter from the nonrelativistic setting is still a very active research area.

The last chapter in the book contains a proof of *existence of the thermodynamic limit*. This refers to the fundamental property that the energy, or for positive temperature systems the *free energy*, per volume is not only bounded but has a limit as the system size tends to infinity. The first proof of this was due to Lieb and Lebowitz [7] and the proof in the book follows this original approach.

Stability of matter may be considered a step towards the more fundamental existence of the thermodynamic limit. Historically this was how stability of matter was viewed [5], but over the decades it has grown to be a subject in its own.

The subject is very much alive. Of particular interest to readers of the book is a recent fairly elementary proof of the Lieb–Thirring inequality which appeared [14, 15] after the publication of the book.

Over the years there have been short reviews on stability of matter (e.g., [6]), and the subject has been treated briefly in mathematical physics texts such as [16, 17]. A comprehensive textbook on the subject useful to researcher and students alike is long overdue. The book *The Stability of Matter in Quantum Mechanics* is just that. A book that an experienced researcher in mathematics or physics can use to learn the subject, a book that the expert in the field must have, and a book that is well suited for a semester course for graduate students. In particular, the book can serve well as an introduction for mathematicians to quantum mechanics.

The book by Lieb and Seiringer presents physical ideas and concepts with mathematical rigor. It is not a book only about mathematics nor a book only about physics. It is a book about both. A book in mathematical physics.

Stability of matter is an advanced subject dealing with complex physical systems and requiring sophisticated mathematics. The book manages to present the material in an easily digestible way. Although basic knowledge of real analysis is required, the book takes great care to aim at a broad audience. What makes the book particularly easy and pleasurable to read is the careful balance between the level of technical details and the clarity and continuity in the line of thought.

The book is written in a style that should be easily accessible to both mathematicians and physicists. I am convinced that it will be an opportunity for many to enter the beautiful subject of stability of matter and all its interesting connections to theoretical physics and pure mathematics.

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