BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 1, January 2013, Pages 153–160
S 0273-0979(2012)01396-4
Article electronically published on October 11, 2012

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DANIEL S. FREED

MR1001453 (90e:57059) 57R55; 58E15, 81E13, 81E40

Atiyah, Michael

Topological quantum field theories.

Institut des Hautes Études Scientifiques. Publications Mathematiques (1988), *no.* 68, 175–186 (1989).

In May 1987 the author [in *The mathematical heritage of Hermann Weyl* (Durham, NC, 1987), 285–299, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988; MR0974342 (89m:57034)] suggested that certain interesting recent developments in the study of low-dimensional manifolds (due to Donaldson, Floer, Gromov, and others) could be profitably formulated and understood in physical "field-theoretic" terms. These suggestions led E. Witten to the discovery of what is now known as "topological field theories" [Comm. Math. Phys. **117** (1988), no. 3, 353–386; MR0953828 (89m:57037)], and the interest these theories have received since then has led to the construction of a large number of models in various (and indeed arbitrary) dimensions.

Now—two years later—the author presents an axiomatic approach to the subject of topological field theories, with the aim of bringing some order into the plethora of theories now known and providing a suitable framework to place these on a mathematically more rigorous footing. The set of axioms is closely related to that put forward by Segal for conformal field theories (formulated in terms of "modular functors"), basically extracting the essential ingredients from the physicists' path-integral approach to quantization of field theories, supplemented by axioms characterizing their "topological" nature. This, despite its simplicity, very efficient and powerful new approach may be expected to lead to new insights and results in the near future.

From MathSciNet, October 2012

Matthias Blau

MR0990772 (90h:57009) 57M25; 17B67, 57N10, 58D15, 58D30, 81E40

Witten, Edward

Quantum field theory and the Jones polynomial.

Communications in Mathematical Physics **121** (1989), *no.* 3, 351–399.

The author introduced the notion of a topological quantum field theory in a previous article [same journal **11** (1988), no. 3, 353–386; MR0953828 (89m:57037)] where he discussed the Donaldson invariants of 4-manifolds. The paper under review interprets the Jones invariants of links in the 3-sphere in terms of quantum field theory and at the same time introduces new invariants of links in arbitrary 3-manifolds. In particular, there are new invariants of closed 3-manifolds. Mathematicians should find this paper more accessible than the article cited above as the field theory here does not involve supersymmetry. We explained some general features of topological quantum field theory in our review of the previous article, so we proceed directly to the current paper.

Fix a compact Lie group G. In this paper the author deals only with simple groups, and to be definite we take $G = \mathrm{SU}(N)$; the general case is discussed further by R. Dijkgraaf and the author ["Topological gauge theories and group cohomology", ibid., to appear], for example. Then if M is an oriented closed 3-manifold and A a connection on a (necessarily) trivial $\mathrm{SU}(N)$-bundle over M, one defines the Chern-Simons invariant $\mathcal{L}(A)$. It takes real values, but changes by integers under gauge transformations. Also, the possible Chern-Simons invariants are parametrized by an integer k. (For a general compact group they are parametrized by $H^4(BG)$.) The variables N and k turn out to be simply related to the variables in the Jones polynomial. The Chern-Simons invariant is the Lagrangian of a classical field theory; the classical solutions are the flat connections. But it is the quantum theory which is of interest. The partition function, defined by integrating $\exp(2\pi i\mathcal{L}(A))$ over the space of connections, is proposed as a new invariant of the 3-manifold M. If C is an oriented loop in M, and R a representation of $\mathrm{SU}(N)$, then for each connection A we can evaluate the character of R on the holonomy of A around C; this is well-defined and gauge-invariant. When this is inserted into the path integral, repeatedly for a link with several components, one gets an invariant of a link in M. The author asserts that since there are no background geometric data (such as a metric) in the theory, these path integrals define topological invariants.

The author first addresses the issue of whether these Feynman path integrals make sense. The usual perturbative calculations of quantum field theory here become the large-k limit of the theory. This relates to previous work of A. S. Shvarts [Lett. Math. Phys. **2** (1978), no. 3, 247–252] and A. M. Polyakov [Modern. Phys. Lett. A **3** (1988), no. 3, 325–328; MR0927055 (89e:81097)]. The leading order behavior is thus computed in terms of the Chern-Simons invariant, Reidemeister torsion, certain combinations of η-invariants, and linking numbers. As expected, these are all topological invariants. This discussion points out one subtlety of the theory—the need to frame the 3-manifolds and the links in order to carry out the path integral. Much more striking is the agreement with the large-k behavior of the exact solutions computed later.

Next, the author considers the path integral on a 3-manifold of the form $\Sigma \times \mathbf{R}$, where Σ is an oriented closed surface. Using standard principles of quantum field theory which relate path integrals to canonical quantization and which prescribe the treatment of symmetries, he is led to the conclusion that the quantum Hilbert space attached to Σ is obtained by quantizing the moduli space of flat $\mathrm{SU}(N)$-bundles over Σ, which is a symplectic manifold. Since this moduli space is compact, the quantum Hilbert space is finite-dimensional. What is the key to the entire paper comes with the realization that this is exactly the description given by G. Segal ["Two-dimensional conformal field theories and modular functors", *IAMP Proceedings* (Swansea, 1988), to appear] of the "space of conformal blocks" in the $(1+1)$-dimensional conformal field theory usually called the Wess-Zumino-Witten model. This space carries a (projective) representation of the mapping class group which has been extensively studied, for example by V. G. Kac and M. Wakimoto [Adv. Math. **70** (1988), no. 2, 156–236; MR0954660 (89h:17036)] in genus 1, and this eventually allows the author to make explicit computations. Similar remarks apply to punctured surfaces, which enter when there are links.

The final ingredient is a general feature of quantum field theories, which we might call the "gluing law". It allows one to calculate a path integral by chopping a manifold into smaller pieces. The author uses it to see how his invariants change under surgery.

At this point one has a concrete prescription for computing the invariants. This prescription is derived from the path integral, and in the author's presentation its validity depends on the path integral, but the algorithm itself is stated in terms of elementary computable formulæ. The author uses this prescription to derive the skein relation of knot theory, and so relate his invariants to the Jones knot polynomials. He also uses it to prove a conjecture of Verlinde (previously proved by G. Moore and N. Seiberg [Comm. Math. Phys. **123** (1989), no. 2, 177–254; MR1002038 (90e:81216)]). Other illustrations of the theory are also given.

The paper ends with a hint that not only the space of conformal blocks of a (1+1)-dimensional conformal field theory, but also the entire (1+1)-dimensional conformal field theory, can be derived from the Chern-Simons theory in 2+1 dimensions.

The author's paper catalyzed much activity by both mathematicians and physicists. By now mathematicians have verified much of what he asserts without using the path integral. The relationship to conformal field theory has been developed in more detail by many physicists. We have neither the space nor the license to do justice to these developments here.

From MathSciNet, October 2012

Daniel S. Freed

MR2335797 (2009b:14051) 14H10; 14F43, 19D06, 55P47

Madsen, Ib; Weiss, Michael

The stable moduli space of Riemann surfaces: Mumford's conjecture.

Annals of Mathematics. Second Series **165** (2007), *no.* 3, 843–941.

In this landmark paper the authors prove Mumford's conjecture: the stable rational cohomology of Riemann's moduli space is a polynomial ring generated by the Miller-Morita-Mumford κ_i-classes of dimension $2i$. By Harer-Ivanov homology stability for the mapping class group, the rational cohomology of moduli spaces $\mathcal{M}_g$ is independent of the genus g in degrees $< (g-1)/2$. Thus Mumford's conjecture gives the rational cohomology of $\mathcal{M}_g$ in these degrees.

More precisely, the authors prove the stronger, homotopy-theoretic version of the conjecture from [I. H. Madsen and U. Tillmann, Invent. Math. **145** (2001), no. 3, 509–544; MR1856399 (2002h:55011)]: a certain map

$$\alpha\colon \mathbb{Z}\times B\Gamma_\infty^+ \xrightarrow{\simeq} \Omega^\infty\mathbb{C}\mathrm{P}^\infty_{-1}$$

is a weak homotopy equivalence. Here Γ_∞ is the stable mapping class group and $B\Gamma_\infty^+$ is its classifying space to which Quillen's plus construction has been applied. The right-hand side is the infinite loop space associated to the Thom spectrum which in degree $n+2$ is the one-point compactification of the canonical n-plane bundle on the Grassmannian manifold of oriented 2-planes in $\mathbb{R}^{n+2}$. Recall that the plus-construction does not change the cohomology of the space and that the mapping class groups have the same rational cohomology as the corresponding moduli spaces. The Mumford conjecture now follows immediately from the well-known rational homotopy equivalences

$$\Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1} \xrightarrow{\otimes L} \Omega^\infty \Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+ \xrightarrow{L} \mathbb{Z} \times B\mathrm{U}$$

under which κ_i corresponds to the integral Chern character class $i!\mathrm{ch}_i$ [cf. op. cit.].

The weak homotopy equivalence α gives much more precise information on the integral cohomology of the stable mapping class group since the cohomology of $\Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1}$ is well-understood. Thus we know the cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients [S. Galatius, Topology **43** (2004), no. 5, 1105–1132; MR2079997 (2006a:57020)] and the divisibility of the κ_i classes in the integral lattice [S. Galatius, I. H. Madsen and U. Tillmann, J. Amer. Math. Soc. **19** (2006), no. 4, 759–779 (electronic); MR2219303 (2006m:57039)].

The philosophy of the proof is to view the homotopy-theoretic Mumford conjecture as an h-principle. While homotopy classes of maps from a manifold X^k to $\mathbb{Z} \times B\Gamma_\infty$ correspond to isomorphism classes of oriented surface bundles, homotopy classes of maps from X^k to $\Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1}$ can be interpreted in terms of certain concordance classes. Indeed, $\Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1}$ arises here as the infinite loop space associated to a Thom spectrum and by standard cobordism theory homotopy classes of maps $X^k \to \Omega^\infty \mathbb{C}\mathrm{P}^\infty_{-1}$ correspond to concordance classes of pairs $(q, \delta q)$ where $q\colon M^{k+2} \to X^k$ is a smooth, proper map of manifolds which is covered by a bundle epimorphism $\delta q\colon TM \times \mathbb{R}^i \to q^*TX \times \mathbb{R}^i$ together with an orientation of the kernel of δq.

The map α assigns to a surface bundle $q\colon M \to X$ the concordance class (q, dq) where dq is the differential of q. Vice versa, to a pair $(q, \delta q)$ one would like to apply Phillips' submersion theorem to replace it by a pair $(\tilde{q}, d\tilde{q})$. However, the submersion theorem applies only when M is open and is false in general. To get around this difficulty, the authors' strategy is to replace M by the open manifold $E := M \times \mathbb{R}$ and to analyse the singularities of the projection $\mathrm{pr}_2\colon E \to \mathbb{R}$ on the fibers of $q \circ \mathrm{pr}_1\colon E \to M \to X$. For this they rely on V. A. Vassiliev's theory [*Complements of discriminants of smooth maps: topology and applications*, Translated from the Russian by B. Goldfarb, Amer. Math. Soc., Providence, RI, 1992; MR1168473 (94i:57020)].

The argument also uses surgery theory and homotopy colimit decompositions. This part of the paper is highly technical and rather complicated. Most of the paper is valid not only for surfaces but for manifolds of any dimension. It is only in the final argument that Harer's homology stability theorem is used in a crucial way.

{Reviewer's remark: A simplified proof of the main theorem is given by the authors, Galatius and the reviewer in ["The homotopy type of the cobordism category", preprint, `arxiv.org/abs/math.AT/0605249`, Acta Math., to appear].}

From MathSciNet, October 2012

Ulrike Tillmann

MR2555928; 2010k:57064 57R56; 18D10, 18G30, 57R15, 57R75

Lurie, Jacob

On the classification of topological field theories.

Current developments in mathematics, 2008, 129–280, *Int. Press, Somerville, MA*, 2009.

The cobordism hypothesis, first proposed by Baez and Dolan in the 1990s, concerns the structure of cobordism categories and is intimately connected to the definition of topological field theory as developed by Atiyah. Lurie's paper presents a development of ideas that led to the cobordism hypothesis in its original statement, then recasts it in many different forms. In fact, the observant reader will find over a dozen theorems labeled "Cobordism Hypothesis" throughout the paper! While the main ideas of the proof are given, it is still as yet an outline with many details still to be developed. However, to call this paper merely a proof sketch would not do it justice. It is also a very readable introduction to the ideas that led to the conjecture, the various stages of development that led to its current form, and the surprisingly many mathematical tools that are used throughout.

This paper is divided into four main sections. The reader wishing to get the main ideas of the cobordism hypothesis and its relevance to topological field theories may be content with reading the first section. The second section develops many of the more complicated definitions that have been stated only informally in the first section. The third section breaks the proof down into five major steps and outlines the main ideas behind them. The fourth and final section presents special cases and generalizations of the cobordism hypothesis.

The starting point for the cobordism hypothesis is the fact that, for any positive integer n, there is a category $\mathrm{Cob}(n)$ whose objects are closed, oriented, smooth manifolds of dimension $n-1$ and whose morphisms are given by diffeomorphism classes of oriented cobordisms between these manifolds. Furthermore, $\mathrm{Cob}(n)$ can be regarded as a symmetric monoidal category under the disjoint union of manifolds. Let $\mathrm{Vect}(k)$ denote the category of vector spaces over a field k, with symmetric monoidal structure given by the tensor product. Then a topological field theory of dimension n is defined to be a symmetric monoidal functor $Z\colon \mathrm{Cob}(n) \to \mathrm{Vect}(k)$. Thus, Z sends disjoint unions of manifolds to tensor products of vector spaces.

In low dimensions, topological field theories can be understood completely. When $n=1$, a topological field theory Z is determined by its value on a single point, partly due to the fact that cobordisms between 0-dimensional manifolds are not terribly complicated, but also because much is determined by the fact that Z is symmetric monoidal. This structure also inherently limits the possible values that Z can take on a point. In particular, this vector space must be finite-dimensional so that it has a well-behaved dual vector space corresponding to the point with opposite orientation. In other words, $\mathrm{Cob}(1)$ is the free symmetric monoidal category generated by a single dualizable object.

The case where $n=2$ is also known; 2-dimensional topological field theories are equivalent to commutative Frobenius algebras. However, in higher dimensions, it becomes harder to understand these functors simply because the manifolds and the cobordisms between them become more complicated. When $n=2$, a complete description is possible because there is a nice way in which we can decompose 2-dimensional manifolds with boundary into pieces which can be easily understood. In higher dimensions, attempting a similar process becomes less feasible. One really needs to consider a higher-categorical version of $\mathrm{Cob}(n)$ and the notion of an extended topological field theory.

Instead of the ordinary category $\mathrm{Cob}(n)$, we utilize an n-category $\mathrm{Cob}_n(n)$, where there are i-morphisms for all $1 \leq i \leq n$. The objects of this n-category are 0-dimensional manifolds, the 1-morphisms are cobordisms between them, the 2-morphisms are cobordisms between the cobordisms, and so forth, up to

n-morphisms, which are given by diffeomorphism classes of cobordisms between lower-dimensional cobordisms.

There are several issues to be considered here. For $n \geq 2$, the cobordisms are often manifolds with corners, and the higher categorical structure that results is not a strict n-category, but a weak one, in which properties such as associativity hold only up to isomorphism and satisfy various coherence conditions. To consider the appropriate generalization of topological field theories, we need an accompanying higher-categorical version of $\mathrm{Vect}(k)$, which in this paper is accomplished simply by allowing the target to be a symmetric monoidal weak n-category $\mathcal{C}$. Again, we need to be concerned with duality, an issue which, not surprisingly, also becomes more complicated in the higher-categorical world. The objects satisfying the necessary conditions are called fully dualizable.

It is in this framework that Baez and Dolan made their version of the cobordism hypothesis, with the additional structure required that all manifolds involved have a framing. They conjectured that $\mathrm{Cob}_n(n)$ is the free weak n-category generated by a fully dualizable object, or that there is a one-to-one correspondence between isomorphism classes of extended $\mathcal{C}$-valued topological field theories Z and isomorphism classes of fully dualizable objects of $\mathcal{C}$.

However, Lurie refines this definition still more. He proves a version of the cobordism hypothesis which is not given in the language of weak n-categories, but in that of (∞, n)-categories, in which there are i-morphisms for arbitrarily large i, but these morphisms are all invertible for all $i > n$. The (∞, n)-categorical version of $\mathrm{Cob}_n(n)$, which Lurie calls Bord_n, has n-morphisms cobordisms, $(n+1)$-morphisms diffeomorphisms between the n-morphisms, $(n+2)$-morphisms isotopies between the diffeomorphisms, and so on up.

Although (∞, n)-categories seem to be more complicated than weak n-categories, they are actually easier to work with, due to a homotopy-theoretic approach that conveniently encodes the weak higher-categorical data. Lurie points out that, although one can recover the original cobordism hypothesis by truncating (∞, n)-categories to weak n-categories, the additional structure is necessary in the proof that he gives.

However, by the end of the first section, (∞, n)-categories have only been defined informally. The second section is primarily concerned with pinning down the precise definition of (∞, n)-categories, functors between them, and fully dualizable objects in them. The major feature that is still not fully addressed is a concrete definition of a symmetric monoidal structure on an (∞, n)-category, but Lurie does refer the reader to his paper which treats the case where $n = 1$ in detail.

For his definition, Lurie takes the n-fold complete Segal spaces as defined by Barwick. When $n = 1$, one recovers Rezk's complete Segal spaces, which are known to be equivalent to other formulations of $(\infty, 1)$-categories. In this way, an (∞, n)-category is an n-fold simplicial object in the category of spaces, satisfying a condition guaranteeing a notion of composition at each level, at least up to homotopy, and a completeness condition relating the objects to morphisms which are equivalences. Using this approach, Bord_n can be defined concretely. This section also gives a precise definition of fully dualizable objects.

Lurie then gives a formulation of the cobordism hypothesis which deals with more general manifolds with structure, not just with oriented or framed manifolds. This version is actually the one which he proves in the third section. He also has some discussion of how one might attempt to prove the cobordism hypothesis for non-smooth manifolds and the difficulties one would expect to have to overcome. In a final subsection, he relates the cobordism hypothesis to work of Galatius, Madsen, Tillmann, and Weiss on the Mumford Conjecture. While this portion may seem like a side note, their methods end up playing a crucial role in the last step of his proof.

The third section is the most technical but also shows the wide range of techniques that must be employed to prove the conjecture. First, the cobordism hypothesis is reduced to an inductive version in which one uses Bord_{n-1} plus some additional information to understand Bord_n. Second, proving it for manifolds with arbitrary structure is reduced to the case of ordinary smooth manifolds. Third, the structure involved in moving from the $(n-1)$ case to the n case is formulated in terms of $(\infty,1)$-categories rather than (∞,n)-categories. The fourth step involves use of Morse theory to understand Bord_n as being built from Bord_{n-1} via handle attachments, which can be interpreted as generators and relations. The fifth step uses obstruction theory to prove that a modified version of Bord_n arising in the previous section is in fact equivalent to the original.

The final section, titled "Beyond the cobordism hypothesis", introduces the reader to ideas that come out of the cobordism hypothesis and its proof. The first subsection is concerned with E_n-structures on (∞,n)-categories and a special case which can be regarded as topological chiral homology. The second subsection looks in some detail at low dimensions, defining Calabi-Yau objects in a symmetric monoidal $(\infty,2)$-category and understanding them using work of Costello. Connections are made with ideas from string topology.

The final two subsections are concerned with generalizations. The first is the case of manifolds with singularities, and the second, which makes use of the first in its proof, is a sketch of the proof of the tangle hypothesis. Also first conjectured by Baez and Dolan, the tangle hypothesis can be considered to be an unstable version of the cobordism hypothesis.

From MathSciNet, October 2012

Julia Bergner

MR2648901 (2011i:57040) 57R56; 18D05, 18D10, 57R58, 81T45

Freed, Daniel S.; Hopkins, Michael J.; Lurie, Jacob; Teleman, Constantin

Topological quantum field theories from compact Lie groups.

A celebration of the mathematical legacy of Raoul Bott, 367–403, *CRM Proc. Lecture Notes, 50*, *Amer. Math. Soc., Providence, RI*, 2010.

The purpose of this paper is to introduce new approaches, in particular those arising from recent developments in higher categorical structures, to the study of topological quantum field theories (TQFTs). In particular, it gives a method for obtaining an extended TQFT (a "0-1-2-3 theory") from a finite group or a torus. Toral theories are of particular interest because, as is shown in this paper, they provide new information about Chern-Simons theory.

The theories of most interest here are those in dimension 3, but the paper works up to this case by first considering 1- and 2-dimensional theories. From there, extended field theories are constructed from finite path integrals, and then applied to the case of a 3-dimensional theory with finite gauge group.

The machinery needed to understand extended TQFTs from a torus group is extensive, beginning with 2-cocycles and continuing with Drinfeld centers of braided tensor categories. Finally, higher algebraic structures, built inductively beginning from complex vector spaces, are needed. These tools allow one to give a quantization procedure for TQFTs and ultimately to give new information about Chern-Simons theory. The tables given at the end of the paper are useful for keeping straight the various theories that arise throughout the paper.

As the authors freely admit in their introduction, there are still many details to be worked out in these constructions, so this paper should be regarded as an overview of many interesting results to be developed in more depth.

From MathSciNet, October 2012

Julia Bergner