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Invariant manifolds and dispersive Hamiltonian evolution equations, by Kenji Nakanishi and Wilhelm Schlag, Zürich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, vi+253 pp., ISBN 978-3-03719-095-1

It often happens that a mathematical subject is beyond the capabilities of contemporary technique for many years followed by a period when a few pioneers develop new techniques and make fundamental progress. Suddenly there is an explosion of interest by many mathematicians, although the original breakthroughs tend to be forgotten. The subject of nonlinear dispersive wave equations is a good example.

In an even broader context, this book is about nonlinear waves. They occur in many branches of science, such as water waves, gas dynamics, laser beams, and quantum field theory. The subject has turned out to be much, much richer than anyone had suspected. The very earliest work, more than 100 years ago, depended heavily on linearization, but that could only illuminate what happens locally, analogous to approximating a surface by its tangent plane at a point. The subject has two main branches. One is the kind where the highest derivatives are nonlinear and there are shock waves. The other is the kind where only the lower-order terms are nonlinear. This book is about the second kind. The very simplest case is a nonlinear term with no derivatives, which appears in various physical applications. Nevertheless, it turns out to be not at all simple.

Typical examples are the nonlinear wave equation (NLW), resp. Schrödinger equation (NLS), generalized Korteweg–deVries equation (gKdV),

$$(1)\quad u_{tt} - \Delta u + f(u) = 0, \qquad iu_t - \Delta u + f(u) = 0, \qquad u_t + u_{xxx} - f(u)_x = 0,$$

where $x \in \mathbb{R}^n$ ($n = 1$ in the third example), Δ is the Laplacian on $\mathbb{R}^n$, and $f(u) = mu + \lambda|u|^{p-1}u$ with $p > 1$ and $m > 0$. In each of these examples the derivative terms tend to make the waves spread out (*disperse*) in the unbounded space $\mathbb{R}^n$ like a linear wave. On the other hand, the nonlinear term $f(u)$ tends to make the waves exaggerate their height differences (*peak*). These two tendencies fight each other. Sometimes one wins, sometimes the other does, and sometimes they balance out.

One might ask why we study very particular PDEs. The answer is that the history of science and mathematics is rife with specific PDEs on which whole theories are based, such as the Cauchy–Riemann equations, the Maxwell equations, and the Navier–Stokes equations.

Fifty years ago almost nothing was known about the global behavior of the solutions (*waves*) of the simple-looking equations (1). Then major discoveries were made in the 1960s and 1970s. During the last decade, there has been an explosion of activity in the mathematical theory. The state of the theory up to 1989 can be found in [3]. It turns out that the qualitative behavior of the waves depends heavily on the sign of λ and on the sizes of p and n. Let us focus on NLW, which is the

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main subject of this book. A big hint comes from the energy

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} \{\tfrac{1}{2}|u_t|^2 + \tfrac{1}{2}|\nabla u|^2 + F(u)\}dx, \qquad F(u) = \tfrac{m}{2}|u|^2 + \tfrac{\lambda}{p+1}|u|^{p+1}, \tag{2}$$

which is independent of time t. In the early years the focus was on classical, say C^2, solutions, but it has turned out that most of the theory is valid assuming merely that $\mathcal{E}(u)$ is finite.

In the more controlled situation, $\lambda > 0$, each term in (2) is positive so that we have bounds on the solution and its first derivatives in the L^2-norm in terms of $\mathcal{E}(u)$. This is called the *defocusing* case, the terminology coming from nonlinear optics. We might reasonably expect a global existence and uniqueness theorem for any initial data. Surprisingly however, this is known only under restrictive conditions, namely, if either $n \le 2$ or $p \le 1 + \frac{4}{n-2}$. On the other hand, for $p > 1 + \frac{4}{n-2}$, only the global existence of weak solutions is known. The idea of the existence proof is simple. One approximates NLW by a problem that is easy to solve, in such a way that the solutions u_k of the approximate problem still have bounded energy, and then one uses weak compactness to show that there is a subsequence of u_k that converges weakly to a solution u of the original problem. But essentially all that we then know about the resulting solution is that its energy $\mathcal{E}(u)$ is bounded by the initial energy. For such a weak solution it is not known whether the energy is independent of time. The solution is so weak that it might not even be a continuous function.

There are several ways to understand the condition on p. For the Sobolev space $H^1(\mathbb{R}^n)$ to be embedded in $L^{p+1}(\mathbb{R}^n)$ requires that $p \le 1+\frac{4}{n-2}$. This means that the last term $\int |u|^{p+1}dx$ in (2) is bounded by the others. Moreover the equation NLW (with $m = 0$) is invariant under the change of scale $u_\alpha(t,x) = \alpha^{2/(p-1)}u(\alpha t, \alpha x)$. Then $\int |\nabla u_\alpha|^2 dx$ scales like α^q where $q \ge 0$ iff $p \le 1 + \frac{4}{n-2}$, meaning that we have control of the very small scales, which is a hint of regularity. The condition $p < 1 + \frac{4}{n-2}$ is called the energy-subcritical case (critical case if equality).

The most fundamental open problem in nonlinear PDEs is whether there is uniqueness—and perhaps regularity—in the supercritical situations where solutions are known to exist but are merely weak, that is, not smooth functions. This problem for NLW has remained open for fifty years. The same problem for the Navier–Stokes equation has been open for eighty years, ever since the work of J. Leray. It is one of the Millenium problems [4] and is connected to the problem of turbulence of a fluid.

Another basic question is, What is the asymptotic behavior as $t \to \pm\infty$? It took many years, but we now know that if $1 + \frac{4}{n} < p \le 1 + \frac{4}{n-2}$, then all the solutions u of NLW *scatter* in the sense that they are asymptotic to linear waves, that is, to solutions $v_\pm$ of the equation with $\lambda = 0$. This means that $\mathcal{E}_0(u(t) - v_\pm(t)) \to 0$ as $t \to \pm\infty$, where $\mathcal{E}_0(u) = \int_{\mathbb{R}^n}\{\frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2\}dx$ is the free energy. The behavior of u in the distant past is given by v_- and in the far future by v_+. The nonlinear mapping from v_- to v_+ is called the scattering operator. The condition that $1+\frac{4}{n} < p$ is equivalent to saying that $\int |u_\alpha|^2$ scales like α^q with $q < 0$. It implies that, for small u, the nonlinear term is small enough that it is asymptotically negligible.

The *focusing* case, $\lambda < 0$, is quite different because some solutions *blow up* in a finite time T^*. In fact, any solution of NLW that satisfies $\mathcal{E}(u) < 0$ does blow up. Negative energy is just a condition on the initial data that the last term in (2) dominates the other two. In fact, as $t \nearrow T^*$, the first two terms in (2) and the last one both become infinite but of opposite sign. On the other hand, if the initial data is small enough, then the solution exists globally (that is, for all times) and scatters because the linear term mu dominates the nonlinear one. On the other hand, if $m = 0$, other interesting situations occur [3], which we will not get into in this review.

Continuing the focusing case for NLW, we see a big gap between the waves that blow up and those that scatter. It turns out that if the energy is not too big, this gap is well understood. A major role is played by the *ground state* Q. Q is the unique positive solution of NLW that does not depend on time (so that $-\Delta Q + f(Q) = 0$) and that minimizes the energy $\mathcal{E}(\phi) = \int_{\mathbb{R}^n}\{\frac{1}{2}|\nabla\phi|^2 + F(\phi)\}dx$ among all functions $\phi(x)$ subject to the constraint $\mathcal{K}(\phi) = \int_{\mathbb{R}^n}\{|\nabla\phi|^2 + \phi f(\phi)\}dx = 0$. It is a nonlinear analogue of the principal eigenfunction of a self-adjoint operator. The existence of a steady state of finite energy requires both focusing ($\lambda < 0$) and energy-subcriticality ($p \leq 1 + \frac{4}{n-2}$). Thus it is another distinguishing feature of the focusing case. It turns out that the ground state Q is a decreasing function of the radial variable $|x|$ and has positive energy. Assuming the subcritical case $p < 1 + \frac{4}{n-2}$ and assuming that the initial data satisfies $\mathcal{E}(u) < \mathcal{E}(Q)$, we have the following dichotomy:

(i) The solution is global and scatters if $\mathcal{K}(u) \geq 0$, while
(ii) the solution blows up if $\mathcal{K}(u) < 0$.

In particular, Q is unstable under some perturbations of the initial data.

Orbital stability is a more subtle issue. In fact, many of the most interesting PDEs are invariant under a group G of transformations. Then G generates a whole family of solutions from a steady state Q. For instance, permitting complex values in NLW and taking $m = 1$, we have the phase transformations $u \mapsto e^{i\theta}u$. Then there is a family (orbit) of *standing wave* solutions $\{e^{i\omega t}Q(x;\omega) : -1 < \omega < 1\}$ of lowest energy that depend on the parameter ω. Orbital stability of one of these standing waves means that $\forall\epsilon > 0,\ \exists\delta > 0$ such that

$$(3)\quad \mathcal{E}\Big(u(0,\cdot) - Q(\cdot;\omega)\Big) < \delta \quad \text{implies} \quad \sup_{0<t<\infty}\ \sup_{\theta\in\mathbb{R}} \mathcal{E}\Big(u(t,\cdot) - e^{i\theta}Q(\cdot;\omega)\Big) < \epsilon.$$

Thus the solution u remains close to the orbit for all future times. It turns out that $e^{i\omega t}Q(x;\omega)$ is orbitally stable if $p < 1 + \frac{4}{n}$ and $|\omega| > (1 - n + 4/(p-1))^{-1/2}$, and it is orbitally unstable otherwise [3].

The three examples in (1) are Hamiltonian, meaning that they can be written in the form $\frac{du}{dt} = J\mathcal{E}'(u)$ with $u(t) \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space and J is a skew-symmetric linear operator. Taking the inner product of any Hamiltonian system with $\mathcal{E}'(u)$, we see that $\mathcal{E}(u(t))$ is independent of time. For the case of NLW, $\mathcal{H} = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, and the Hamiltonian form is

$$(4)\qquad \partial_t \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + f(u) \\ \dot{u} \end{pmatrix}.$$

For NLS, J is multiplication by i and $\mathcal{E}(u) = \int_{\mathbb{R}^n}\{\frac{1}{2}|\nabla u|^2 + F(u)\}dx$. For KdV, $J = \partial_x$ and $\mathcal{E}(u) = \int_{\mathbb{R}}\{\frac{1}{2}|u_x|^2 + F(u)\}dx$. The various particular wave equations have a lot in common but there are important differences as well and most problems demand separate analyses.

What happens to waves, solutions of the focusing NLW, $\lambda < 0$, whose energies are greater than the energy of Q? This is almost completely open territory. What the authors do in this book is to show that the situation is tremendously complicated even if the energy is only a tiny bit greater. In particular, they investigate in great detail the waves that begin initially in a small neighborhood of Q. Throughout most of the book they assume that $p = n = 3$ and that the solutions are radial (depend only on t and $r = |x|$). Thus the equation is $u_{tt} - \Delta u + u - |u|^2 u = 0$. Since they are working near Q, the linearized operator $L = -\Delta + 1 - 3Q^2$ plays a central role. It has exactly one negative eigenvalue, which is a hint that there might be solutions of NLW lurking near Q that blow up. In fact, (4) can be rewritten as

$$\partial_t \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \mathcal{L} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} + \begin{pmatrix} 0 \\ 3Qu^2 + u^3 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} 0 & 1 \\ -L & 0 \end{pmatrix}. \tag{5}$$

Now the linear matrix operator $\mathcal{L}$ has exactly one negative and one positive eigenvalue while all the rest of its spectrum (including the continuous spectrum) is imaginary.

Here is the book's main conclusion: Starting with any initial data of energy less than $\mathcal{E}(Q) + \epsilon^2$ for small ϵ, the solution of NLW either (a) blows up or (b) scatters to zero or (c) scatters to $\pm Q$. Furthermore, there exist radial solutions such that any one of these three scenarios occurs as $t \to +\infty$ and any other one as $t \to -\infty$. So this set of solutions splits up into *nine nonempty subsets*! By "scatter to zero" is meant "scatter" in the sense mentioned earlier in this review. By "scatter to Q" is meant that $u(t,x) - Q(x)$ scatters.

Furthermore, among the waves that begin near Q, those that satisfy (a) or (b) for positive times leave the vicinity of Q. The spectrum of $\mathcal{L}$ tells us about the behavior of these solutions as, say, $t \to +\infty$. Namely, the unstable direction, associated with the positive eigenvalue of $\mathcal{L}$, is the tangent at Q to a smooth local curve in $\mathcal{H}$, called the *unstable manifold* W^u. The stable direction, coming from the negative eigenvalue of $\mathcal{L}$, leads to another curve, the *stable manifold* W^s, for which $u(t) \to Q$ as $t \to +\infty$. The complementary initial data, for which the solution scatters to Q but does not converge to Q, form the *center manifold* W^c, which has codimension two. The concept of stable/center/unstable manifolds is exactly the same as in dynamical systems. Among these three manifolds, the center manifold is the most subtle because a small nonlinear perturbation could in principle kick the solution towards stability or instability.

A novel ingredient of their proof is their "one-pass" theorem, which states that a solution that begins near Q can enter or exit a neighborhood of Q at most once. In fact, a solution in the neighborhood that is not on the center-stable manifold moves away from Q because of the unstable mode associated with the positive eigenvalue of $\mathcal{L}$. Then another global identity, related to scaling in x, is used to show that it continues to move away. In fact, one considers the family of dilations $D_\alpha : u(x) \to u(\alpha x)$ and its generator $x \cdot \nabla$, symmetrizes it, and multiplies the equation by the result to get the virial (Morawetz) identity

$$\partial_t \int_{\mathbb{R}^n} u_t \, \tfrac{1}{2}(x \cdot \nabla u + \nabla \cdot (xu)) dx = \int \{|\nabla u|^2 - \tfrac{3}{4}|u|^4\} dx =: \mathcal{K}_2(u).$$

The signs of $\mathcal{K}_2$ and $\mathcal{K}$ are used in conjunction with a classical argument [2] to show that the solution stays away from Q.

A natural problem that remains open is to understand the eventual behavior of the *super-energetic* focusing waves, those with energies greater than that of the ground state. Interesting numerical computations in [1] indicate that the boundary of the set $\mathcal{S}_+$ of all the waves that scatter to zero appears to be very irregular.

The nonradial case is outlined in Section 6.1. This is more difficult because instead of the two stationary solutions $\pm Q$, there is a whole manifold $\mathcal{M}_Q$ of solutions generated from Q by spatial translations and Lorentz transformations. The extra parameters of $\mathcal{M}_Q$ have to be controlled in the construction of the center-stable manifold that emanates from $\mathcal{M}_Q$.

This book is written by two leaders in the theory of nonlinear waves. It is particularly well written and organized. There are very clear preambles and summaries of the individual chapters. After an introductory chapter, there is an analysis of the waves below the ground state energy. The core of the book is Chapters 3–5, which treats the slightly larger energies for radial NLW with $n = p = 3$. In Chapter 3 they also discuss the linear dispersive estimates in the analogous NLS case, and this is reconsidered in Section 6.3. They have a section on the one-dimensional NLW and another one on the energy-critical NLW for $m = 0$ and $p = 1 + \frac{4}{n-2}$. The book is an expanded version of the authors' research papers that is based on a course given by one of them.

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Walter A. Strauss

Brown University, Providence, RI 02912

E-mail address: `wstrauss@math.brown.edu`