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SELECTED MATHEMATICAL REVIEWS

related to the papers in the previous section by
ELENA FUCHS and ALEX KONTOROVICH

MR0230204 (37 #5767) 50.10
Coxeter, H. S. M.
The problem of Apollonius.
The American Mathematical Monthly **75** (1968), 5–15.

This is the author's Presidential Address to the Canadian Mathematical Congress (28 August, 1967) [reprinted from Canad. Math. Bull. **11** (1968), 1–17]. He begins with an elegant proof (due in essence to P. Beecroft [Lady's and Gentleman's Diary **139** (1842), 91–96]) of the Descartes-Soddy relation between the radii of four mutually tangent circles: $2\sum(1/r^2) = (\sum 1/r)^2$.

He then points out that of three non-intersecting circles, either one or none separates the other two. In the former case there is no (real) circle tangent to all three, in the latter (when he calls them an Apollonian triad) eight. Two non-intersecting circles have an inversive-invariant "distance", most simply defined (though not by the author, who confines himself severely to real geometry) as the absolute magnitude of their (pure imaginary) angle of intersection at an imaginary common point. The "distances" by pairs of three non-intersecting circles satisfy the triangle law if and only if they are an Apollonian triad.

Every Apollonian triad can be inverted either into three congruent circles (i.e., all with the same radius) or into three homothetic circles (i.e., with two common tangent lines, all three being in the same angular region between these).

Two non-intersecting circles have a unique "mid-circle", with respect to which they are inverse. The "mid-circles" by pairs of an Apollonian triad are coaxal.

From MathSciNet, January 2013

P. Du Val

MR1913879 (2003f:00005) 00A05; 20H10, 28A80, 30F40, 37F30, 52C26, 68R15
Mumford, David; Series, Caroline; Wright, David
Indra's pearls. (English)
Cambridge University Press, *New York*, 2002. xx+396 *pp*. $50.00.
ISBN 0-521-35253-3

Felix Klein, one of the great nineteenth-century geometers, discovered in mathematics an idea prefigured in Buddhist mythology: the heaven of Indra contained a net of pearls, each of which was reflected in its neighbour, so that the whole Universe was mirrored in each pearl. Klein studied infinitely repeated reflections and was led to forms with multiple co-existing symmetries, each simple in itself, but whose interactions produce fractals on the edge of chaos. For a century these images, which were practically impossible to draw by hand, barely existed outside the imagination of mathematicians. However in the 1980s the authors embarked on the first computer exploration of Klein's vision, and in so doing found further extraordinary images of their own. The book is written as a guide to actually coding the algorithms which are used to generate the delicate fractal filigrees, most of which have never appeared in print before.

The first two chapters, The language of symmetry, and A delightful fiction, contain material on Euclidean symmetries and complex numbers, respectively, introducing as clearly as possible and with complementary graphics the mathematical terminology which is used throughout the book. Chapter 3, Double spirals and Möbius maps, introduces the basic double spiral maps, called Möbius symmetries, on which all of the later constructions rest. From then on, the authors build up ever more complicated ways in which a pair of Möbius maps can interact, generating more and more convoluted and intricate fractals. In Chapter 4, The Schottky dance, the Schottky groups are introduced and a way of drawing all Schottky circles is described. In Chapter 5, Fractal dust and infinite words, a new limit set program based on the principle of depth-first, rather than breadth-first, search is proposed. Chapter 6, Indra's necklace, is devoted to the quasi-Fuchsian groups. Here the program is adapted to draw quasicircles to receive the so-called Indra's pearls. The construction of the Apollonian gasket is considered in Chapter 7, The glowing gasket. The problem of choosing parameters is discussed in Chapter 8, Playing with parameters. The progression through the book is the investigation of more and more remarkable ways in which two Möbius maps can "dance" together. Chapter 9, Accidents will happen, investigates a special collection of groups called "accidents". These are the beautiful double cusp groups in which two symmetries are forced to be parabolic, corresponding to two different rational numbers of one's own choice. These groups lie right on the borderline between the relatively well-behaved quasi-Fuchsian regime and the total disorder of non-discreteness. This is not the whole story. The group is said to be singly degenerate if the quasicircle has swallowed one-half of the ordinary set and doubly or totally degenerate if it has swallowed both halves. These stranger groups hovering on the same boundary between order and chaos are considered in Chapter 10, Between the cracks. In Chapter 11, Crossing boundaries, the authors actually reach the frontiers of current research—Kleinian groups. The entire development is summarized in the last Chapter 12, Epilogue, and in the Road Map on the final page.

Beginners can learn to understand what the images mean and follow the step-by-step instructions for writing computer programs that generate them. Experts in the geometry of discrete groups can see how the images relate to ideas that take them to the forefront of research.

From MathSciNet, January 2013

Vasily A. Chernecky

MR1971245 (2004d:11055) 11H55

Graham, Ronald L.; Lagarias, Jeffrey C.; Mallows, Colin L.; Wilks, Allan R.; Yan, Catherine H.

Apollonian circle packings: number theory.

Journal of Number Theory **100** (2003), *no.* 1, 1–45.

Four mutually tangent circles of an Apollonian circle packing have curvatures that satisfy the Descartes equation

$$2(x^2 + y^2 + z^2 + w^2) = (x + y + z + w)^2.$$

The authors treat this as a Diophantine equation by considering packings in which all the curvatures are integers. Within each such packing is a smallest "root" configuration of four circles from which all other quadruples can be generated by a certain finitely generated subgroup of $\mathrm{GL}(4,\mathbb{Z})$. They give an exact formula for $N^*_{\mathrm{root}}(-n)$, the number of primitive root quadruples containing a given negative integer $-n$, by relating this to a certain class number. They are able to count the number of integral solutions to height T by relating this to a question for the Lorentz form $-W^2+X^2+Y^2+Z^2$ and hence to known formulas for $r_3^*(m)$, the number of primitive representation of m as a sum of three squares. They consider the set of integers representable by a packing and show that these satisfy certain congruence conditions modulo 24 (or modulo 12 for certain packings). They conjecture that all sufficiently large integers satsifying these conditions appear in a given packing, a result analogous to Zaremba's conjecture about continued fractions of rationals. They support this by a heuristic density argument. The paper is very clearly written and covers a lot of ground. It has points of contact with numerous aspects of number theory and is well worth reading.

From MathSciNet, January 2013

D. Boyd

MR2173929 (2009a:11090a) 11E57; 11H55, 52C26

Graham, Ronald L.; Lagarias, Jeffrey C.; Mallows, Colin L.; Wilks, Allan R.; Yan, Catherine H.

Apollonian circle packings: geometry and group theory. I. The Apollonian group.

Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science **34** (2005), *no.* 4, 547–585.

MR2183489 (2009a:11090b) 11E57; 11H55, 52C26

Graham, Ronald L.; Lagarias, Jeffrey C.; Mallows, Colin L.; Wilks, Allan R.; Yan, Catherine H.

Apollonian circle packings: geometry and group theory. II. Super-Apollonian group and integral packings.

Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science **35** (2006), *no.* 1, 1–36.

MR2183490 (2009a:11090c) 11E57; 11H55, 52C26

Graham, Ronald L.; Lagarias, Jeffrey C.; Mallows, Colin L.; Wilks, Allan R.; Yan, Catherine H.

Apollonian circle packings: geometry and group theory. III. Higher dimensions.

Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science **35** (2006), *no.* 1, 37–72.

In this series of papers, which follows [R. L. Graham et al., J. Number Theory **100** (2003), no. 1, 1–45; MR1971245 (2004d:11055)], written by largely the same set of authors, the authors are motivated by the study of Apollonian packings, which is to say, arrangements of mutually tangent circles (or, in higher dimension, balls). Such objects have been studied for over two thousand years, starting (unsurprisingly) with Apollonius, continuing through Descartes (whose theorem, generally known as "Soddy's formula", after Soddy, who expressed it as a poem some two hundred and fifty years after its discovery by Descartes), on to Koebe and Schottky (who were interested in such objects from the viewpoint of groups of conformal transformations generated by reflections), to Thurston (who used patterns of tangent circles in his proof of geometrization for a large class of 3-manifolds; see, e.g., J. W. Morgan's survey [in *The Smith conjecture (New York, 1979)*, 37–125, Academic Press, Orlando, FL, 1984; MR758464]). In higher dimensions, packing of balls was studied by Daryl Cooper and the reviewer in [Math. Res. Lett. **3** (1996), no. 1, 51–60; MR1393382 (97k:52022)]; in fact, some of the observations made by the authors in the papers under review were made by Cooper and Rivin, apparently unbeknownst to the authors.

In any event, the Apollonian packing studied by the authors is quite specific; it is one where you start with the "Descartes' configuration"—a set of four mutually tangent circles (some of which might be straight lines), and then generate further circles from inversions in the four circles you started with. This corresponds to the limit set of a Coxeter group (essentially) obtained by reflections in the facets of a right-angled ideal octahedron in three-dimensional hyperbolic space. This object (call it A) is unique up to Möbius transformation, but the authors are interested in those Möbius transformations μ for which all of the curvatures of the resulting arrangement of packings are integral (there is also an even stronger condition on integrality of "circle-curvature" coordinates introduced by the authors, which is fascinating but too cumbersome to be described in this review). It is not a priori obvious that there are any such, but in fact, once one is found, the set of such packings is in one-to-one correspondence with an arithmetic subgroup of the Möbius group (which the authors refer to as the Lorentz group $\mathrm{O}(3,1)$); for a hyperbolic geometer like this reviewer, $\mathrm{Isom}(\mathbb{H}^3)$ is the natural identification. This arithmetic group preserves the Descartes (or Soddy) quadratic form; a variant also preserves what the authors call the Wilker quadratic form (after J. B. Wilker). Using this, identification of the work of W. Duke, Z. Rudnick and P. C. Sarnak [see Duke Math. J. **71** (1993), no. 1, 143–179; MR1230289 (94k:11072)] can be used to count how many integral packings there are with the "coordinates" of a given set of circles being bounded by some constant. For related deep observations see `www.math.princeton.edu/sarnak/AppolonianPackings.pdf`.

The authors discuss a number of diverse and interesting results, and the papers (which, together with the earlier paper [R. L. Graham et al., op. cit.], form a mid-length monograph—it might have been wiser to publish the papers in the form of such a monograph) are written beautifully and are a joy to read. Since the papers are of an exploratory character, it is difficult to point to a specific "main theorem", and the interested reader of this review is advised to go and read the papers—they are worth the time.

From MathSciNet, January 2013

Igor Rivin

MR2813334 (2012d:11072) 11D09; 11E16, 11E20, 52C15

Bourgain, Jean; Fuchs, Elena

A proof of the positive density conjecture for integer Apollonian circle packings.

Journal of the American Mathematical Society **24** (2011), *no.* 4, 945–967.

A Descartes configuration consists of four mutually touching circles with distinct tangents, and recursively adding circles tangent to three of the already constructed circles yields an *Apollonian circle packing*. These are completely described by the original four circles in the Descartes configuration, and the curvatures (a, b, c, d) of these circles satisfy the relation

$$2\left(a^2 + b^2 + c^2 + d^2\right) - (a + b + c + d)^2 = 0$$

(a theorem of Descartes from 1643). The main theorem in the paper under review settles a positive-density question of R. L. Graham et al. [J. Number Theory **100** (2003), no. 1, 1–45; MR1971245 (2004d:11055)]. Namely, if we denote by $\kappa(P, X)$ the number of distinct integers up to X occurring as curvatures in the integer Apollonian circle packing P, then there exists a constant $c > 0$ depending on P such that

$$\kappa(P, X) \geq cX.$$

From MathSciNet, January 2013

Matthias Beck