BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 4, October 2013, Pages 663–674
S 0273-0979(2012)01391-5
Article electronically published on December 6, 2012

Integral geometry and Radon transforms, by Sigurdur Helgason, Springer, New York, 2010, xiv+301 pp., ISBN 978-1-4419-6054-2, hardcover

1. Integral geometry

The book under review is an excellent introduction to the group theoretical and analytic aspects of the field by one of its pioneers. Before reviewing the book, we will provide an overview of the field.

Integral geometry draws together analysis, geometry, and numerical mathematics. It has direct applications in PDEs, group representations, and the applied mathematical field of tomography. The fundamental problem in integral geometry is to determine properties of a function f in the plane or three-dimensional space or other manifolds from knowing the integrals of f over lines, planes, hyperplanes, spheres, or other submanifolds.

The history of integral geometry starts in the early 1900s with Radon, Funk, Lorenz, and others. In 1917, Johann Radon [60] proved an inversion formula for what has become known as the *classical Radon line transform:* if ℓ is a line in the plane and f is an integrable function, then

$$R_L f(\ell) = \int_{x\in\ell} f(x)ds, \tag{1.1}$$

where ds is the arc length measure on the line ℓ. This is the mathematical model behind X-ray tomography, which we discuss in Section 5. One can extend this to the *hyperplane transform*, R_H, which integrates over all hyperplanes H in $\mathbb{R}^n$. Radon also proved an inversion formula for R_H in $\mathbb{R}^3$. (According to Bockwinkel [5], the inversion for R_H had already been obtained by H. A. Lorenz before 1906.) In 1936, Cramer and Wold [10] proved injectivity of the hyperplane transform for probability measures. In 1938, Fritz John [46] used PDEs to characterize the range of certain Radon transforms, and he proved uniqueness results and inversion formulas. Gelfand [24, 25], Helgason [39], and others in the 1960s developed inversion methods, range theorems, and other properties of these transforms.

Integral geometers study transforms that integrate functions over a wide range of sets. In 1916, Paul Funk [17] built upon earlier work of Minkowski [54] and obtained inversion formulas for the so-called Funk transform which integrates even functions over great circles on the sphere S^2. More generally, one can consider the *X-ray transform* on a Riemannian manifold M, which integrates a suitable function over geodesics in M, and the *spherical mean operator* on M, which takes the average of f over geodesic spheres.

In the late 1950s, Gelfand and Graev [22] obtained relations between the harmonic analysis on semisimple Lie groups and Radon transforms on horospheres, which are translates of orbits of maximal unipotent groups. Along with the pioneering work of Harish-Chandra [36, 37], this laid the foundation for harmonic analysis on semisimple Lie groups and symmetric spaces.

Motivated by these examples, we define the generalized Radon transform as follows. Let X and Ξ be smooth manifolds, and assume every $\xi \in \Xi$ is associated

2010 *Mathematics Subject Classification.* Primary 44A12, 43A85, 53C65.

with a smooth closed submanifold $\widehat{\xi} \subset X$, all with the same dimension. Given a smooth measure dm_ξ on each $\widehat{\xi}$ we can define a Radon transform integrating functions in X over the manifolds $\widehat{\xi}$:

$$Rf(\xi) = \int_{x\in\widehat{\xi}} f(x) dm_\xi(x). \tag{1.2}$$

For this transform to be defined for all $f \in C_c(X)$, the manifolds $\widehat{\xi}$ and measures dm_ξ must satisfy certain restrictions. In addition, for the transform R to be invertible, the manifolds $\widehat{\xi}$ must "fill out" X sufficiently well and the measures dm_ξ need to be sufficiently "nonzero". With this in mind, here are fundamental questions for R.

- When is a Radon transform, R, injective? If R is injective, is there a closed form inversion formula? If not, what is its null space?
- What is the range of R?
- What are the mapping properties of R? Is the transform continuous between appropriate function spaces? Is the range closed? If R is invertible, is its inverse continuous (and in what topologies)?
- How does the support of a function relate to the support of its Radon transform?
- What does R do to the singularities of a function?

All of these questions have motivated deep mathematics and are important for applications in mathematics and science, as we discuss in Section 5. Professor Helgason has proved important theorems answering most of these questions, and they are thoroughly addressed in the book under review.

In this review, we will outline some developments in integral geometry from Radon and Funk to the present, including the double fibration and the group setting (Section 2), harmonic analysis (Section 3), microlocal analysis and what R does to singularities (Section 4), and some applications to tomography (Section 5). Finally, we will describe Professor Helgason's book and put it in context of the field. We will not address the version of integral geometry related to convexity and geometric probability (which is sometimes referred to as *classical integral geometry*), although there is a rich literature in this area also (see e.g., [18, 61] for introductory treatments and references).

2. Double fibrations and homogeneous spaces in duality

In integral geometry, the relation between the manifolds X and Ξ leading to the Radon transform (1.2) above arises from an *incidence relation* between points in X and Ξ, given as follows. Suppose that Z is a regular submanifold of $X \times \Xi$ such that the natural projections p and π

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ X & & \Xi \end{tikzcd} \tag{2.1}$$

are fibrations. The diagram (2.1) is called a *double fibration*. Often, p is assumed to be a proper map. Points $x \in X$ and $\xi \in \Xi$ are said to be *incident* iff $(x, \xi) \in Z$. The submanifold Z itself is called an *incidence relation*. For each $\xi \in \Xi$, let $\widehat{\xi} = \{x \in X \mid x \text{ is incident to } \xi\}$. In integral geometry, each $\widehat{\xi}$ is assumed to be a regular submanifold of X, and is assigned a suitable measure (arising from the geometry of X and Ξ) over which we integrate functions on X to obtain the Radon transform.

The notion of an incidence relation was first formulated in the early 1940s by Chern [8] in a group-theoretic setting. Chern observed that certain important formulas by Crofton and Blaschke in classical integral geometry could best be understood and generalized in terms of the incidence relation associated to a pair X and Ξ of homogeneous spaces of the same Lie group G. This incidence relation was later observed by Helgason [40] to be the most suitable framework by which to study Radon transforms on homogeneous spaces of a Lie group G. Helgason called this notion of incidence and the associated double fibration diagram *homogeneous spaces in duality*. This framework for integral geometry was generalized by Gelfand, Graev, and Shapiro [23] to pairs of manifolds without any group actions. Most Radon transforms which commute with the action of a Lie group, and certainly all the transforms in the book under review, fall under the rubric of homogeneous spaces in duality. It is therefore appropriate to review its basic elements.

Suppose that X and Ξ are now homogeneous spaces of a Lie group G, i.e., $X = G/K$ and $\Xi = G/H$, where H and K are closed subgroups of G. Let x and ξ be points in X and Ξ, respectively, with $x = gK$ and $\xi = \gamma H$. Following Chern's definition, we say that x and ξ are *incident* if they intersect as cosets in G. If we put $L = K \cap H$, then the incidence relation Z in (2.1) can be identified with the homogeneous space G/L, and the double fibration above corresponds to the diagram

(2.2)
$$\begin{tikzcd} & G/L \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ X = G/K & & \Xi = G/H, \end{tikzcd}$$

where again p and π are the natural projection maps.

For simplicity, let us now assume that the groups G, H, K, and L are all unimodular. The natural left invariant measures on the corresponding homogeneous spaces will then be denoted by dg_K, dh_L, etc.

If $\xi = \gamma H$ is a point in Ξ, then $\widehat{\xi} = \{\gamma hK : h \in H\}$, so $\widehat{\xi}$ is a left translate of the orbit $\widehat{\xi}_0 = \{hK : h \in H\}$ of H in X. This orbit is diffeomorphic to the homogeneous space H/L. We assign the natural H-invariant measure dh_L on H/L to this orbit, and then we can left-translate this measure in a well-defined way to each $\widehat{\xi}$. Assuming that the orbit $\widehat{\xi}_0$ (and hence each translate $\widehat{\xi}$) is closed in X, the measures on the $\widehat{\xi}$ now permit us to define the Radon transform Rf of a suitable function f on X. Group theoretically, Rf is the function on Ξ given by

(2.3)
$$Rf(\gamma H) = \int_{H/L} f(\gamma\, hK)\, dh_L.$$

Analogously, we can also define a *dual transform* R^* which integrates a function φ on Ξ over the submanifolds $\check{x} = \pi(p^{-1}(x))$:

(2.4)
$$R^*\varphi(gK) = \int_K \varphi(gkH)\, dk_L.$$

The operators R and R^* are linear maps and are formal adjoints in the sense that

$$\int_\Xi Rf(\xi)\,\varphi(\xi)\,d\mu(\xi) = \int_X f(x)\,R^*\varphi(x)\,dm(x),$$

for f and φ suitable functions on X and Ξ, respectively, where $d\mu(\xi) = dg_H$ and $dm(x) = dg_K$, suitably normalized.

While this framework is too general to even guarantee invertibility, the obvious G-equivariance of the Radon and its dual transform allows one to study them in the context of the interplay between the harmonic analysis on the homogeneous spaces G/K and G/H. With additional conditions on G, H, and K, one can obtain more properties for R and R^* using group theory.

As a specific example, let us consider the Funk transform. The geodesics on S^2 are great circles and thus correspond to planes through the origin on $\mathbb{R}^3$. Therefore, the Funk transform is a map from functions on S^2 to functions on $\mathbb{RP}^2$. The group SO(3) acts transitively on both spaces, and it is not hard to see that this transform is the one corresponding to the double fibration (2.2), with $G = \mathrm{SO}(3)$ and $K = \mathrm{SO}(2)$ the subgroup fixing the north pole $(0, 0, 1)$. Then, H is the subgroup consisting of the matrices

$$\begin{pmatrix} \det\sigma & 0 \\ 0 & \sigma \end{pmatrix} \qquad (\sigma \in \mathrm{O}(2))$$

fixing the x-axis. Here the incidence relation between points $x \in S^2$ and great circles $\xi \in \mathbb{RP}^2$ given by the double fibration (2.2) is the usual inclusion relation.

As another example, consider the classical Radon transform R_H on $\mathbb{R}^3$ from Section 1. Both $\mathbb{R}^3$ and the manifold Ξ of 2-planes in $\mathbb{R}^3$ are homogeneous spaces of the group G of rigid motions of $\mathbb{R}^3$, and R_H is the Radon transform associated with the double fibration (2.2), where K and H are the subgroups of G fixing the origin and the (x, y)-plane, respectively. The incidence relation between points and planes is again the usual one of inclusion.

3. Harmonic analysis and PDEs

There are significant relations between Radon transforms and harmonic analysis, as well as associated subjects such as representation theory and partial differential equations. Below we will limit ourselves to providing a few basic examples.

A fundamental observation first obtained by Radon is the relation between the Fourier transform $\mathcal{F}$ and the transform R_H on $\mathbb{R}^3$. Each plane in $\mathbb{R}^3$ can identified with an ordered pair

$$(\omega, p) \cong \{x \in \mathbb{R}^3 \,:\, x \cdot \omega = p\}, \tag{3.1}$$

for some $\omega \in S^2$ and $p \in \mathbb{R}$. By integrating a suitable function f along planes orthogonal to ω, we see that its Fourier transform $\widehat{f}$ satisfies

$$\widehat{f}(s\omega) = \int_{-\infty}^{\infty} R_H f(\omega, p)\, e^{-ips}\, dp. \tag{3.2}$$

The relation above, called the *Fourier Slice Theorem*, can be generalized to other transforms which integrate functions over d-planes in $\mathbb{R}^n$. It is used to invert the Radon transform (see formula (5.1) below) and to describe its range. Gelfand and Graev [22] extended the projection slice theorem to invert the horospherical transform, a key component in their study of the representation theory (via principal series) and harmonic analysis of complex semisimple Lie groups.

More generally, using the Iwasawa decomposition $G = NAK$, Helgason introduced an analogue of the Fourier transform (now called the *Helgason Fourier transform*) on a noncompact Riemannian symmetric space G/K [38, 41]. This Fourier transform satisfies a Fourier Slice Theorem with respect to the horocycle Radon transform, and the interplay between these transforms has properties similar to their counterparts in $\mathbb{R}^n$. For instance, the horocycle Radon transform can be inverted using the inversion formula for the Fourier transform. The Fourier Slice Theorem can be construed as a starting point for much of the study of the harmonic analysis on G/K and on the space G/MN of horocycles. For example, the Poisson transform on G/K is just the restriction of the dual horocycle transform to an appropriate space of distributions (or other functionals) on G/MN. Likewise, intertwining operators between spherical principal series representations of G can be obtained by convolutions with conical distributions on G/MN [41].

There is much cross-fertilization between Radon transforms and partial differential equations. For example, one of the classical solutions to the initial value problem for the wave equation in $\mathbb{R}^n$ uses the hyperplane Radon transform [47]. Another example is the ultrahyperbolic equation of Fritz John [46], which is used to characterize the range of the X-ray transform and the related *parametric Radon transform* on $\mathbb{R}^3$, defined by the integral

$$R_P f(x, y; \alpha, \beta) = \int_{-\infty}^{\infty} f(x + t\alpha, y + t\beta, t)\, dt,$$

where f is a suitable function on $\mathbb{R}^3$ and $(x, y; \alpha, \beta) \in \mathbb{R}^4$. One can easily show that $R_P f$ vanishes under the action of the second-order operator

$$\frac{\partial^2}{\partial x \partial \beta} - \frac{\partial^2}{\partial y \partial \alpha}.$$

It turns out that the kernel of this operator is precisely the range of the parametric transform. We refer the reader to the book by Gelfand, Gindikin, and Graev [21] for an introductory treatment of parametric Radon transforms, in which one finds, for instance, the Gauss hypergeometric transform expressed as a parametric line transform of certain monomials with complex powers. One can find another treatment in Ehrenpreis's book [11].

Systems of differential equations can also be used to characterize the ranges of other types of Radon transforms. For example, the Asgeirsson mean value theorem [2, 47] shows that the Darboux equation may be thought of as characterizing the range of the spherical mean operator on $\mathbb{R}^n$. As another example, the joint eigenfunctions of the G-invariant differential operators on a symmetric space G/K are precisely the images under the Poisson transform of hyperfunctions on the boundary of G/K. This is the *Helgason conjecture*, which Helgason proved in 1970 for the hyperbolic plane [41], and which was proved in general by six Japanese mathematicians (Kashiwara, Kowata, Minemura, Okamoto, Oshima, and Tanaka) in 1978 [49].

For yet another example, consider the Cauchy problem for the wave equation on $\mathbb{R}^n$. One of the classical solutions involves the "shifted" dual to the Radon transform [47]. This solution has been extended by Helgason [43] to multitemporal systems on symmetric spaces, and it is particularly appealing because it makes apparent Huygens' principle in the even multiplicity case.

Radon and dual transforms associated with double fibrations of the type described above are compelling objects of study precisely because they intertwine the left regular representations of G on spaces of functions, distributions, differential forms, bundle sections, etc., on X and Ξ. This G-equivariance may be used, for instance, to diagonalize and invert the transform [31, 48, 64] or to characterize its range or support [28, 29]. This equivariance has also been used to study cusp forms and theta series in number theory [27].

4. Generalized Radon transforms and microlocal analysis

Group invariant Radon transforms have a beautiful theory, as outlined in the previous sections and described in the book under review. In this section, we discuss one theme in integral geometry that is not related to groups.

Microlocal analysis is, in the broadest terms, the study of singularities of functions and distributions, and how Fourier integral operators transform these singularities. In particular, Guillemin and Sternberg [33, 34] proved that, under certain assumptions on the double fibration and with a nowhere zero smooth measure, the Radon transform R is an elliptic Fourier integral operator. Under a specific assumption on the Radon transform, the *Bolker assumption*, Guillemin proved that R^*R is an elliptic pseudodifferential operator so R^*Rf reproduces all singularities of the function f [33, 34]. Guillemin originally showed that the Radon hyperplane transform and the point-horocycle transform on rank one symmetric spaces satisfy this assumption [32]. Subsequently, Quinto calculated the top-order symbol of R^*R [57] and Beylkin determined the top- and lower-order symbols of an operator related to R^*R that is motivated by applications [4].

If R does not satisfy the Bolker assumption, then R^*Rf can have more singularities than f or singularities of f can be masked. This is shown in the seminal article [30] for the X-ray transform on admissible line complexes in manifolds. New classes of Fourier integral operators (so called $I^{p,l}$ classes [35]) have been used to describe how, for such singular Radon transforms, R, the composition R^*R adds singularities (e.g., [30]). Although injectivity is difficult to prove without a group structure, analytic microlocal analysis has been used to prove injectivity and support theorems for transforms with nowhere zero real-analytic measures on hyperplanes [6, 7], geodesics, and other submanifolds [16, 51].

5. Tomography

Tomography is one of the most practical and useful applications of integral geometry, and it has motivated beautiful pure and applied research. We will mention a few areas that are most closely related to the previous sections and to the book under review.

The best known type of tomography is X-ray tomography (X-ray CT), which is modeled by the Radon line transform, R_L defined in equation (1.1). X-ray CT was introduced to the larger mathematical community in the late 1970s in part through introductory articles by Smith, Solmon, and Wagner [63] and Shepp and Kruskal [62]. Then, in 1979, Allan Cormack and Godfrey Hounsfield won the Nobel Prize in Medicine for their pioneering work creating X-ray CT scanners and developing algorithms for their machines. The standard inversion algorithm for R_L is filtered backprojection

$$f = \frac{1}{4\pi} R_L^* \sqrt{-d^2/dp^2} R_L f, \tag{5.1}$$

where p is the coordinate given in (3.1) [55]. This formula is easy to implement and gives good images for planar X-ray CT data when X-rays are taken over lines uniformly distributed through the object. It can be proven using the Fourier Slice Theorem (equation (3.2)).

Many tomography problems use *limited data*. That is, some data are missing and the standard inversion algorithms, which require complete data, cannot, in general, be used.

One type of limited data occurs in nondestructive evaluation of large objects such as rocket shells. X-rays will not penetrate the thick center of a rocket. However, they will penetrate the outer annulus of the object because it is thinner. This data, over lines that do not meet the center of the object, is called *exterior data*. The classical support theorem for the Radon line transform (see Theorem 6.1 in the next section) shows that one can reconstruct the outside annulus of an object from exterior data, and inversion methods exist (e.g., [9]). However, inversion is highly unstable, which can be seen from estimates [53] as well as an argument of Finch [14] showing that the inverse is continuous in no range of Sobolev norms. In contrast, inversion of R_L with complete data is continuous of order 1/2 in Sobolev norms [55] and so is only mildly ill-posed.

Another type of limited data occurs in micro-CT when one wants to image a small part of an object, such as a single organ in the body or a small part of an industrial object. This part of the object is called the *region of interest*, and the data over lines meeting that region are called *region of interest data*.

The range theorem for R_L, Theorem 6.2, characterizes the range of this transform in terms of moment conditions. This theorem can be used to construct a function in the null space for region of interest data. Despite the presence of a null space, all singularities of the object inside the region of interest are visible from this data, which follows from microlocal analysis (e.g., [58]).

Microlocal analysis can be used to understand what singularities are "visible" in limited data problems in X-ray tomography [58, 15] and radar [56]. Singularity detection algorithms have been developed for the region of interest problem in planar X-ray CT [12, 65] cone beam CT (e.g., [50, 66]), as well as other modalities (e.g., [59]). Understanding how singularities are added can help one understand and mitigate the effect of these added singularities (e.g., [13, 15]).

The mathematics we have discussed comes up in other problems in tomography. The model in single photon emission tomography [52] involves a Radon line transform with a measure that is not group invariant. Models in radar [56], sonar, and thermoacoustic tomography [1] involve circular, elliptical, or spherical mean transforms and their generalizations, and researchers use harmonic and microlocal analysis, PDEs, and group theory to develop reconstructions methods and properties. There is much elegant mathematics in these areas, but the details are beyond the scope of this review.

The Range, Support, and Fourier Slice Theorems are important to integral geometry and tomography, and the book under review introduces the mathematics behind them.

6. The book

Integral geometry is an important subject in the large field of geometric analysis, and this very readable book serves as an essential introduction to the topic. Throughout the book, the group-theoretic point of view, which the book's author helped to introduce, is emphasized. The wide range of examples in the book presented serves to demonstrate the power and effectiveness of the use of group techniques.

The first chapter of the book introduces the classical Radon and d-plane transforms, which integrate a function on $\mathbb{R}^n$ over affine hyperplanes and d-dimensional planes, respectively. Inversion formulas are proved for these transforms, and the author establishes the results and themes (given in Section 1) which recur throughout the field. Among the two most important results are the original Support Theorem for Radon transforms as well as the Range Theorem. Recall that R_H is the classical Radon hyperplane transform.

Theorem 6.1 (The Support Theorem [25, 39])**.** *Suppose that $f \in C(\mathbb{R}^n)$ satisfies the following conditions:*

(i) *For each integer $k > 0$, $|x|^k f(x)$ is bounded.*
(ii) *There exists a constant $A > 0$, such that $R_H f(\omega, p) = 0$ whenever $|p| > A$.*

Then $f(x) = 0$ for $|x| > A$.

The rapid decrease condition, (i), in the Support Theorem is sharp; if it is weakened, then the conclusion is false.

Theorem 6.2 (The Range Theorem [25, 39])**.** *The Radon transform R_H is a linear bijection from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto the vector space $\mathcal{S}_H$ of smooth rapidly decreasing functions φ on the space of affine hyperplanes in $\mathbb{R}^n$ satisfying the following moment conditions: for any nonnegative integer k, there exists a homogeneous degree k polynomial P_k on $\mathbb{R}^n$ such that*

$$\int_{-\infty}^{\infty} \varphi(\omega, p)\, p^k\, dp = P_k(\omega).$$

While the use of groups is already hinted at in the first chapter of the book, the formal group-theoretic double fibration (given in Section 2 above) is presented in the second chapter. Numerous examples are provided: among these are the Funk and horocycle transforms on the hyperbolic plane $\mathbb{H}^2$, various integral transforms on Grassmannians, and theta series and cusp forms. Even the Poisson transform on the unit disk,

$$Pf(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2}\, f(e^{i\theta})\, d\theta,$$

is shown to be a Radon transform between functions on homogeneous spaces of $\mathrm{SU}(1, 1)$.

In the third chapter, the author considers inversion and support theorems for various integral transforms on two-point homogeneous spaces. These include the transform which integrates a function over all totally geodesic submanifolds of a given dimension, the horocycle transform on rank one noncompact symmetric spaces, and the antipodal transform on a rank one compact symmetric space.

The fourth chapter deals with aspects of the X-ray transform on symmetric spaces, including inversion formulas, support theorems, and injectivity results. The results include an inversion formula which the author recently proved for the X-ray transform on a compact symmetric space.

Chapters IV and V deal with orbital integrals on isotropic Lorenz spaces and mean value operators, respectively, including a group-theoretical proof of Fritz John's iterated mean-value formula, and a proof of Asgeirsson's mean value theorem for rank one symmetric spaces.

The last three chapters of the book constitute a long and helpful appendix containing the basic material needed by a beginning reader. These include a rapid introduction to Fourier transforms, Lie groups, and the geometry of symmetric spaces. These chapters make the book essentially self-contained.

The author's two previous books on the subject [42, 44], which appeared in 1980 and 1999, are already considered classics in the field. This book adds many more results and examples, including a fuller treatment of integral transforms on constant curvature spaces, and a full proof of the inversion formula for the antipodal transform on a rank one compact symmetric space. Further significant improvements over the two previous editions are the inclusion of helpful exercises at the end of each chapter, as well as the addition of the extensive appendix mentioned above.

Beginning graduate students and interested nonspecialists will gain from the book because of its clear exposition and comprehensive nature. Practitioners of integral geometry will find it a valuable reference with complete and clear proofs as well as specialized items of interest, such as orbital integrals, generalized Riesz potentials, and the group-theoretical basis for inversion formulas using shifted dual transforms. Because of its group-theoretical emphasis, this book does not include topics for which the reliance on groups is less important or which require a more specialized background. Among such excluded topics are integral transforms on differential forms [19]; the κ operator and universal inversion formulas [19, 21, 26]; the Penrose transform and the relation of integral geometry to twistor theory and cohomology [3]; the relation of integral geometry to representation theory ([20] or [45]); Radon transforms and microlocal analysis [33]; and computed tomography [55]. Rather than aiming to be comprehensive, the book focuses on important topics in integral geometry which any beginner in the field ought to know, and it presents the material in a lucid and appealing fashion.

The author of this book is one of the pioneers of integral geometry, and his mathematics has deeply influenced the pure and applied parts of the field. This book is a well-written and beautiful introduction to integral geometry from the perspective of group actions. It has valuable thought provoking exercises. It demonstrates the richness of the subject and provides new examples as well as clear and complete proofs of the fundamental theorems in the field. The book answers questions posed at the start of this review for the most important classical Radon transforms. It should be read by anyone who would like to learn more about integral geometry.

Acknowledgments

The authors thank Sigurdur Helgason for his mentorship and inspiration throughout the years. The authors also thank Jens Christensen, Peter Kuchment, and Gestur Ólafsson for valuable comments about this review. The second author was partially supported by NSF grant DMS-0908015.

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