BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, January 2014, Pages 169–172
S 0273-0979(2013)01424-1
Article electronically published on July 12, 2013

Felix Hausdorff—Gesammelte Werke. Vol. IA, *Allgemeine Mengenlehre*, U. Felgner, V. Kanovei, P. Koepke, and W. Purkert (Editors), Springer, Berlin, 2013, xxvi+537 pp., ISBN 3-642-25598-4

Felix Hausdorff—Gesammelte Werke. Vol. III, *Deskriptive Mengenlehre und Topologie*, U. Felgner, H. Herrlich, M. Hušek, V. Kanovei, P. Koepke, G. Preuß, W. Purkert, and E. Scholz (Editors), Springer, Berlin, 2008, xxii+1005 pp., ISBN 3-540-76806-7

Felix Hausdorff—Gesammelte Werke. Vol. VIII, *Literarisches Werk*, F. Volljardt and U. Roth (Editors), Springer, Berlin, 2010, xx+877 pp., ISBN 3-540-77758-8

Felix Hausdorff—Gesammelte Werke. Vol. IX, *Korrespondenz*, W. Purkert (Editor), Springer, Berlin, 2012, xx+729 pp., ISBN 3-642-01116-0

Another four volumes of the edition of Hausdorff's collected works have now appeared, and they maintain the handsome standards set by the earlier volumes. The earlier volumes covered some Hausdorff's work on set theory in Volume II, including the first edition of his book *Mengenlehre*; his work on analysis, algebra, and number theory in Volume IV; astronomy, optics, and probability theory in Volume V; and his philosophical work in Volume VII. (They were reviewed in this journal, see (Gray 2007).)

The recent volumes include Volume IA, which contains just over 500 pages on general set theory, and Volume III, which is almost twice the size, and covers descriptive set theory and topology. These topics loom large in Volume IX, which carries Hausdorff's correspondence, and Volume VIII is devoted to Hausdorff's literary work.

As the editors rightly say at the start of Volume IA, most mathematicians know Hausdorff's name, and for many his ideas are as necessary as the air they breathe, yet few know the full extent and significance of his work. Their original plan was for Volume I to contain Hausdorff's work on ordered sets and some other topics, together with a biography. That biography has grown and is still in progress, so they have brought forward the mathematical works. Eight papers on ordered sets are presented, together with editorial commentaries, and a further eighty pages of material is taken from the *Nachlass*, including Hausdorff's incisive review of Russell's *Principles of Mathematics*.

Among the issues here are Hausdorff's proof that there are $2^{\aleph_0}$ different types of countable linear orderings, which Hausdorff discovered independently of Bernstein in 1901. It follows from this result that the set of such countable linear orderings cannot be an example of a cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$ so we are in the territory of the continuum hypothesis. The editor's comment that Bernstein's result shows that $\aleph_1 \leq 2^{\aleph_0}$ is a slip, if $\aleph_1$ is defined as the smallest uncountable cardinal. Subsequent papers by Hausdorff include his paper "Grundzüge einer Theorie der geordneten Menge" of 1908 in which he showed that every linearly ordered set is a union of a dense set and a scattered (*zerstreut*) set (a set is scattered if it has no subset dense in itself and it admits no order preserving embedding of the rational

2010 *Mathematics Subject Classification.* Primary 01A60.

numbers). He went on to show that all scattered sets can be constructed and all dense sets classified, and proved the theorem that every linearly ordered set is either scattered or an ordered sum of scattered sets with a densely ordered indexing set. Another particularly noteworthy paper is the one from 1936 on gaps that the editor, Vladimir Kanovei, describes as one of the most valuable research articles of the pre-forcing era of set theory.

The centrepiece of Volume III is the reprint of the second edition of Hausdorff's book *Mengenlehre*, first published in 1927 and reprinted here from the third edition of 1935. A fascinating introductory essay by Kanovei and Walter Purkert, who has worked on the edition from its inception, observes that the second and third editions differ markedly from the first. Hausdorff left out the material on measure and integration theory, much of the material on ordered sets, and the applications to Euclidean spaces. Moreover, he restricted his attention to metric spaces, which were being intensively studied at the time. The result was that he gave an account of descriptive set theory that was a great influence, especially on Russian and Polish mathematicians. The book discusses cardinal numbers, order types and ordinal numbers, Borel sets and Suslin sets, the Borel hierarchy, and then many of the most fundamental aspects of metric space theory from an abstract point of view that embraces Fréchet's L-spaces, the relationship between point sets and order types, and the theory of real functions.

The editors trace the origin of descriptive set theory back to the so-called descriptive theory of functions begun by Baire, Borel, and Lebesgue in 1899. They were concerned with when mathematical objects may be said to exist, and they required that such objects be definable, or characterisable, in a finite number of words. Lebesgue's "Sur les fonctions représentables analytiquement" (1906) is the founding document for the field—such was the opinion of Alexandrov and Hopf—and one of Hausdorff's most important contributions was to recast the theory in terms of sets rather than functions. He extended this work in a series of papers reproduced here that were published in *Fundamenta Mathematicæ* in the 1930s.

The subject was further advanced by Luzin in 1930, when he published his *Leçons sur les ensembles analytiques et leurs applications*. The editors quote from an enthusiastic review by von Neumann, who hailed the work, and also Suslin's contributions, as "one of the most important steps in point set theory" alongside Hausdorff's earlier contribution. As the editors show, the two books are rather different, the chief point of overlap being Borel sets. The editors also include a number of reviews of Hausdorff's *Mengenlehre*. Of at least equal interest is the preface Alexandrov and Kolmogorov wrote for the Russian translation, which is given here in a German translation, where they point out the masterly account that Hausdorff gave of Suslin's A-sets.

Alexandrov had been asked by Luzin in 1915 to tackle the continuum hypothesis in the context of what has become known as the Borel hierarchy, and he was able to show that every uncountable set at each level contains a perfect set, and therefore has the cardinality of the continuum. The same result was obtained by Hausdorff. In 1917 Suslin showed that Alexandrov's methods extended to some sets that were not Borel sets, and which he called A-sets, to distinguish them from Borel or "B" sets. Lebesgue then favoured the term "analytic sets", and in the mid-1930s a vicious priority dispute broke out among the Russians in the context of the Luzin affair. (The only full-length account of this momentous event, which mixed mathematics, religion, and Stalinist politics, is Demidov and Levshin (1999) in Russian; Roger Cooke is preparing an English translation.) As the editors quietly notice, Hausdorff preferred the term Suslin sets.

Volume IX, *Korrespondenz*, opens with over 130 pages of letters between Hausdorff and Alexandrov and Urysohn. The letters begin in 1923 and continue to 1935. Unusually, the editors have been fortunate to have much of both sides of the exchange. The correspondence began with Alexandrov and Urysohn writing to Hausdorff to say they planned to study his work on topology. As Alexandrov wrote later (see p. 9), the Russians differed from Hausdorff in finding the new mathematics more interesting than the logical analysis of classical theory favoured by Hausdorff.

There are some forty pages of letters to Friedrich Engel, which are interesting because Lie's work was an early interest of Hausdorff's, and the Campbell-Baker–Hausdorff theorem is an important one in the theory of Lie algebras (Hausdorff's paper of 1906 was included in Volume IV). There are numerous brief exchanges with many others, including Bieberbach, Blaschke, Carathéodory, Courant, Hilbert, Lie, von Mises, Threlfall, Vietoris, and Zermelo. The bulk of the correspondence consists of letters from Hausdorff; the editor, Walter Purkert, has concluded that the replies must be considered lost as a result of the war.

Each correspondent is given a short introduction, and the letters are amply commentated. For example, in his letter to Hausdorff of 10 February 1939 Courant apologised that he could not help Hausdorff but expressed the hope that Weyl might be able to. Purkert then describes what was done to get Hausdorff a position in America, and quotes letters from Weyl, von Neumann, and Courant. The ten surviving letters from Hausdorff to Hilbert, which span the period from 1900 to 1932, enter into topics in the foundations of geometry, number theory, and editorial matters concerning *Mathematische Annalen*, as well as some set theory. In the last letter, written just after Hilbert turned 70, Hausdorff wrote that "As the title of *princeps mathematicorum* is already taken, I would propose to call you *dux mathematicorum* were it not that the names *dux*, *duce*, *Führer* have become so discredited by people, [and] I would rather call you *lux mathematices*."

The forty pages of letters to Fritz Mauthner, the author and philosopher, remind us that Hausdorff was not only a mathematician, but a philosopher, critic, and playwright. Volume VIII contains Hausdorff's literary work, which he published under the assumed name of Paul Mongré. The book *Ekstasen*, originally published in 1900, contains his many short poems, and it is followed by fourteen of Mongré's essays on cultural topics. For the first time in these volumes, the editors' commentaries are longer than the original texts and sometimes overwhelm them, which may be a measure of how much we have forgotten about Hausdorff's life and times. The complementary part of Hausdorff's work in philosophy occupies Volume VII.

When the long-awaited biography of Hausdorff, now scheduled for Volume IB, and Volume VI (*Geometrie, Raum und Zeit*) are published, the long journey to restore all Hausdorff's work to the light and describe the life of this important figure of his time will be over. We can then begin properly to appreciate all that he did, but the editors and the publisher may be thanked already for producing an edition that in its own way matches the achievements of this diverse and creative mathematician and intellectual.

Acknowledgment

The author thanks Roger Cooke for his comments on earlier drafts of this review.

References

[1] S. S. Demidov and B. V. Levshin (eds.), *Delo akademika Nikolaya Nikolaevicha Luzina* (The case of Academician Nikolai Nikolaevich Luzin), Russkii Khristianskii Gumanitarnyi Institut, St Petersburg, 1999.

[2] J. J. Gray, Review of Hausdorff, F. *Gesammelte Werke* vols. II, IV, V, and VII, *Bull. Amer. Math. Soc.* **44** (2007), 471–474.

[3] F. Hausdorff, *Gesammelte Werke*, Vol. II, *Grundzüge der Mengenlehre*, Springer, Berlin, 2002; Vol. IV, *Analysis, Algebra, und Zahlentheorie*, Springer, Berlin, 2001; Vol. V, *Astronomie, Optik, und Wahrscheinlichkeitstheorie*, Springer, Berlin, 2006; Vol. VII, *Philosophisches Werk*, Springer, Berlin, 2004.

Jeremy Gray

Department of Mathematics and Statistics, Open University
Milton Keynes MK7 6AA, United Kingdom
E-mail address: j.j.gray@open.ac.uk