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The ambient metric, by Charles Fefferman and C. Robin Graham, Annals of Mathematics Studies, Vol. 178, Princeton University Press, Princeton, New Jersey, 2012, x+113 pp., ISBN 978-0-691-15313-1

In Einstein's theory of general relativity one starts with a Lorentzian metric $\widetilde{g}$ defined on a suitable manifold and constructs from it the Ricci tensor $\widetilde{R}$, in terms of which the Einstein field equations may be written. Until very recently, in a vacuum these equations were that $\widetilde{R}=0$ (now modified by a tiny but positive *cosmological constant* seemingly responsible for the accelerating expansion of the Universe).

Be that as it may, the Ricci tensor and the *Ricci-flat* Lorentzian metrics $\widetilde{g}$ (those with $\widetilde{R}=0$) are objects of independent mathematical interest. The Ricci tensor is already remarkable in its being a tensor! In local coordinates its expression is a complicated mess of partial derivatives that somehow conspires to be independent of choice of coordinates. Well, some coordinates might be better than others, and the Fefferman–Graham ambient metric construction is based on the astonishingly good things that happen in coordinates (t,x^i,ρ) for which a Lorentzian metric happens to take the form

$$\widetilde{g}_{IJ}=\begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 g_{ij} & 0 \\ t & 0 & 0 \end{pmatrix}, \tag{1}$$

where $g_{ij}=g_{ij}(x,\rho)$ as ρ varies is a one-parameter family of Riemannian metrics in the coordinates $(x^1,x^2,\ldots,x^n)$. The uninterested reader may safely skip the following details and re-enter the discussion at Theorem 1.

As the authors put it, a "straightforward but tedious" calculation shows that, if we denote differentiation with respect to ρ by $'$, write g^{ij} for the inverse of g_{ij}, and employ the Einstein summation convention, then

$$\widetilde{R}_{IJ}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \widetilde{R}_{ij} & \frac{1}{2}g^{kl}(\nabla_k g'_{il}-\nabla_i g'_{kl}) \\ 0 & \frac{1}{2}g^{kl}(\nabla_k g'_{jl}-\nabla_j g'_{kl}) & -\frac{1}{2}g^{kl}g''_{kl}+\frac{1}{4}g^{kl}g^{pq}g'_{kp}g'_{lq} \end{pmatrix},$$

where

$$\widetilde{R}_{ij}=\rho g''_{ij}-\rho g^{kl}g'_{ik}g'_{jl}-\tfrac{1}{2}\rho g^{kl}g'_{kl}g'_{ij}-\tfrac{n-2}{2}g'_{ij}-\tfrac{1}{2}g^{kl}g'_{kl}g_{ij}+R_{ij} \tag{2}$$

and R_{ij} is the Ricci tensor of $g_{ij}(x,\rho)$ for ρ fixed. Suppose we look for Ricci-flat $\widetilde{g}_{IJ}$, dubbing such creatures *ambient metrics*. Setting $\rho=0$ in (2) implies that

$$\left[\tfrac{n-2}{2}g'_{ij}+\tfrac{1}{2}g^{kl}g'_{kl}g_{ij}\right]_{\rho=0}=R_{ij}\big|_{\rho=0},$$

equivalently that

$$g'_{ij}(x,0)=\frac{2}{n-2}\left(R_{ij}-\frac{R}{2(n-1)}g_{ij}\right)\Big|_{\rho=0},$$

where $R\equiv g^{ij}R_{ij}$ is the scalar curvature of g_{ij}. At this point, those familiar with conformal differential geometry will recognise the tensor on the right-hand side of

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this equation as twice the *Schouten tensor* P_{ij}, equivalent to the Ricci tensor but more congenial with respect to conformal transformations: if $\hat{g}_{ij} = \Omega^2 g_{ij}$ for some positive smooth function Ω, then $\hat{\mathrm{P}}_{ij} = \mathrm{P}_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j - \frac{1}{2} g^{kl} \Upsilon_k \Upsilon_l g_{ij}$, where $\Upsilon_i = \nabla_i \log \Omega$ and ∇_i is the Levi-Civita connection for g_{ij}. Thus, we come to the following conclusion.

Theorem 1. *If a Lorentzian metric of the form*

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x, \rho)\, dx^i dx^j$$

is Ricci-flat, then

$$g_{ij}(x, \rho) = g_{ij}(x) + 2\mathrm{P}_{ij}(x)\rho + O(\rho^2)$$

where $g_{ij}(x) \equiv g_{ij}(x, 0)$ and $\mathrm{P}_{ij}(x)$ is the Schouten tensor of the metric $g_{ij}(x)$.

One is naturally led to ask about the higher order terms in ρ, and a remarkable link with conformal differential geometry begins to emerge as follows.

Theorem 2. *If $g_{ij}(x)$ is conformally flat, then the Lorentzian metric*

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x, \rho)\, dx^i dx^j$$

is flat when

$$g_{ij}(x, \rho) = g_{ij}(x) + 2\mathrm{P}_{ij}(x)\rho + g^{kl}(x)\mathrm{P}_{ik}(x)\mathrm{P}_{jl}(x)\rho^2.$$

Notice that we are not assuming that $g_{ij}(x)$ is flat, only that $\Omega^2(x) g_{ij}(x)$ is flat for some positive smooth function $\Omega(x)$. Evidence that something deep is afoot begins to accumulate:

Theorem 3. *If $g_{ij}(x)$ is an Einstein metric, that is to say $\mathrm{P}_{ij}(x) = \lambda g_{ij}(x)$ for some λ, necessarily constant, then*

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x, \rho)\, dx^i dx^j$$

is Ricci-flat when

$$g_{ij}(x, \rho) = g_{ij}(x) + 2\mathrm{P}_{ij}(x)\rho + g^{kl}(x)\mathrm{P}_{ik}(x)\mathrm{P}_{jl}(x)\rho^2.$$

A noteworthy aspect of Theorems 2 and 3 is that the expansion of $g_{ij}(x, \rho)$ in ρ terminates at second order. In general, this is not the case but, concerning second order terms, we find:

Theorem 4. *If a Lorentzian metric of the form*

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x, \rho)\, dx^i dx^j$$

is Ricci-flat and $n \neq 4$, then

$$g_{ij}(x, \rho) = g_{ij}(x) + 2\mathrm{P}_{ij}(x)\rho + [g^{kl}(x)\mathrm{P}_{ik}(x)\mathrm{P}_{jl}(x) - \tfrac{1}{n-4} B_{ij}(x)]\rho^2 + O(\rho^3),$$

where $B_{ij}(x)$ is the so-called Bach tensor*, a classical trace-free symmetric 2-tensor manufactured from second covariant derivatives of the curvature.*

Theorem 4 is a relatively straightforward deduction from (2): differentiate once with respect to ρ and set $\rho = 0$. Clearly, something goes astray when $n = 4$ and, more generally, a formal power series solution exists to all orders only when n is odd. In even dimensions, the expansion breaks down at order $n/2$. In all dimensions, explicit formulæ for higher order terms are formidable.

So, what's going on here? Firstly, if $g_{ij}(x, \rho)$ in (1) is the standard Euclidean metric η_{ij} on $\mathbb{R}^n$ independent of ρ, then the Lorentzian metric $\widetilde{g}_{IJ}$ is flat and arises geometrically by inverse stereographic projection $\mathbb{R}^n \hookrightarrow S^n$ together with realising this sphere as the space of light rays through the origin in $\mathbb{R}^{n+2}$ with its standard Lorentzian metric. Equivalently, the change of coordinates $(t, x^i, \rho) \rightsquigarrow (s, x^i, r)$ for $\rho < 0$, defined by setting $-2\rho = r^2$ and $s = rt$ gives

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2\eta_{ij}\, dx^i dx^j = s^2\frac{dr^2 + \eta_{ij}\, dx^i dx^j}{r^2} - ds^2,$$

which one recognises as the metric cone over hyperbolic $(n+1)$-space realised by the standard Poincaré metric

$$\frac{dr^2 + \eta_{ij}\, dx^i dx^j}{r^2} \text{ on } \{(x^i, r) \in \mathbb{R}^{n+1} \mid r > 0\}.$$

These identifications reek of conformal geometry, especially when one bears in mind that the Lorentzian symmetry group $\mathrm{SO}^{\uparrow}(n+1,1)$ is also both isometric motions of hyperbolic $(n+1)$-space and conformal motions of the round n-sphere (equivalently, the one-point conformal compactification of Euclidean n-space via stereographic projection). In other words, the special form of metric (1) certainly fits well with flat conformal geometry.

But, as a consequence of Theorem 1, this link with conformal geometry persists into the curved setting as follows. Suppose we just ignore the higher order terms and, for any Riemannian metric $g_{ij}\, dx^i dx^j$ on a smooth manifold M, consider the Lorentzian metric

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2(g_{ij} + 2\mathrm{P}_{ij}\rho)\, dx^i dx^j \tag{3}$$

suggested by Theorem 1, where P_{ij} is the Schouten tensor of g_{ij}. This Lorentzian metric $\widetilde{g}_{IJ}$ induces its own Levi-Civita connection $\widetilde{\nabla}_I$ on the tangent bundle in the (t, x^i, ρ) variables, and we may pull back this connection to the space

$$\{(t, x^i, \rho) \mid \rho = 0\} = \{(t, x^i) \mid t > 0\} = \mathbb{R}_+ \times M. \tag{4}$$

Furthermore, the substitution $t \mapsto e^{\lambda}t$ for constant λ simply rescales (3) by $e^{2\lambda}$, having no effect on the Levi-Civita connection $\widetilde{\nabla}_I$. Consequently, the pulled-back connection on $\mathbb{R}_+ \times M$ descends to a connection on a certain rank $n+2$ vector bundle $\mathcal{T}$ on M. A computation shows that this connection may be written as

$$\begin{bmatrix} \sigma \\ \mu_j \\ \tau \end{bmatrix} \longmapsto \begin{bmatrix} \nabla_i\sigma - \mu_i \\ \nabla_i\mu_j + g_{ij}\tau + \mathrm{P}_{ij}\sigma \\ \nabla_i\tau - g^{jk}\mathrm{P}_{ij}\mu_k \end{bmatrix},$$

where ∇_i is the metric connection on M for g_{ij}. This formula is very familiar from conformal differential geometry. It was introduced by T. Y. Thomas in 1926 and is now seen as equivalent to the conformal Cartan connection, introduced on the level of frame bundles by É. Cartan in 1923. In summary, Theorem 1 somehow incorporates one of the most basic constructions in conformal differential geometry, namely the Cartan connection.

We have yet to make any formal link between Lorentzian metrics of the form (1) and the conformal class of the metric g_{ij} on M. Such a link is initiated by a geometric interpretation of the space (4) as the *bundle of scales* $\mathcal{G}$ of a conformal manifold $(M, [g])$. The point is that a conformal manifold is an equivalence class of Riemannian metrics on M, where two metrics g and $\widehat{g}$ are said to be equivalent whenever $\widehat{g} = e^{2\Upsilon}g$ for some smooth function Υ on M. Otherwise expressed, this equivalence is precisely between metrics that measure the same angles according to the usual formula $\cos\theta = g_{ij}X^iY^j/\sqrt{g_{ij}X^iX^jg_{kl}Y^kY^l}$. There is no preferred metric on a conformal manifold. The best one can do is to gather the metrics from the conformal class into a bundle $\pi : \mathcal{G} \to M$. Any particular metric trivialises this bundle $\mathcal{G} = \mathbb{R}_+ \times M$ and then, for $(s(x), x)$ a section of $\mathbb{R}_+ \times M$, the corresponding metric is s^2g (and it is convenient to scale with s^2 so that s has the units of length). Thus, a naive interpretation of Theorems 1, 4, and higher order expansions, obtained by iteratively solving (2) for a power series expansion of $g_{ij}(x, \rho)$, is as building an *ambient space* $\widetilde{\mathcal{G}} \supset \mathcal{G}$ with an *ambient metric* $\widetilde{g}$ characterised chiefly by its being Ricci-flat. It may be reasonably compared with trying to solve the wave equation from characteristic initial data.

Making all this precise, at the same time freeing up the discussion from any choice of coordinates, is a formidable task. This is what is accomplished in the first four chapters of this excellent monograph, *The ambient metric*. Thus, it is the "straight pre-ambient metrics" that may be placed in the normal form

$$2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_{ij}(x, \rho)\, dx^i dx^j$$

and may then be directly related (in Chapter 4) to asymptotically hyperbolic "Poincaré metrics". When the dust settles, the ambient metric construction sees only the conformal class of the initial metric $g_{ij}(x, 0)$ and provides a powerful tool in the local theory of conformal differential geometry. More on this shortly. . . .

This monograph has been keenly anticipated for more than 25 years! The original article [C. Fefferman and C. R. Graham, "Conformal invariants", in *Élie Cartan et les Mathématiques d'Aujourd'hui*, Astérisque **1985**, Numéro Hors Série, 95–116], initiated a revolution in the theory of local invariants on a conformal manifold just as [C. Fefferman, *Parabolic invariant theory in complex analysis*, Adv. in Math. **31** (1979), 131–262] had done on CR manifolds. Now, at last, all details are available, and more besides.

As previously stated, the first four chapters carefully set up the construction itself. The remaining five chapters are concerned with applications, of which there are many. As already hinted, to first order in ρ the ambient metric construction is equivalent to the Cartan connection, which is often regarded as the final goal in local conformal differential geometry. Indeed, having constructed the Cartan connection, it is often stated that one has "solved the equivalence problem" and, sure enough, the Cartan connection is flat if and only if the conformal class contains the flat metric (one says that a metric in this class is "conformally flat"). But the Cartan connection is only the first step into the conformal world. The ambient metric assembles higher order information in a subtle but usable fashion. Perhaps the most striking objects to emerge are intimately related:

- the Fefferman–Graham obstruction tensor,
- Branson's Q-curvature,
- the GJMS (Graham–Jenne–Mason–Sparling) operators.

These are concepts in even-dimensional conformal differential geometry and all are concerned with the breakdown in the ambient metric at order $n/2$ when n is even. As the name suggests, the obstruction tensor $\mathcal{O}_{ij}$ is exactly what prevents the coefficient of $\rho^{n/2}$ in the power series for $g_{ij}(x, \rho)$ from being determined (and even when $\mathcal{O}_{ij} = 0$, the coefficient is not uniquely determined so the ambient metric is still in trouble). It is a conformally invariant symmetric trace-free tensor and coincides with the classical Bach tensor when $n = 4$. Although explicit formulæ for

$\mathcal{O}_{ij}$ are out of the question except in low dimensions, it has many fine properties: like the Bach tensor it is divergence-free and vanishes if g_{ij} is conformally Einstein. Branson's Q-curvature is a high order n-form-valued Riemannian invariant. When $n = 4$, it is

$$\big[-\nabla^i\nabla_i\mathrm{P} - 2\mathrm{P}^{ij}\mathrm{P}_{ij} + 2\mathrm{P}^2\big]d\mathrm{vol}_g, \text{ where } \mathrm{P} \equiv g^{ij}\mathrm{P}_{ij}.$$

It is not conformally invariant, but if $\widehat{g} = e^{2\Upsilon}g$, then

$$\widehat{Q} = Q + P_n\Upsilon,$$

where the GJMS operator P_n is a self-adjoint linear conformally invariant operator from functions to n-forms of the form $\Upsilon \mapsto dSd\Upsilon$, for some differential operator S from 1-forms to $(n-1)$-forms. In particular, Q transforms by a divergence from which it follows that $\int_M Q$ is a conformal invariant. To close this circle of ideas, if h_{ij} is any trace-free symmetric tensor, then it turns out that

$$\int_M Q_{g+\epsilon h} = \int_M Q + \epsilon\int_M \mathcal{O}_{ij}h^{ij} + O(\epsilon^2).$$

These are delicate results for which the ambient metric is essential since, for higher even n, explicit formulæ are extreme (notwithstanding that Fefferman and Graham have recently used the ambient metric construction to establish some surprising recurrence relations due to Juhl, which give some handle on these formulæ).

These matters are discussed in Chapter 7. Before that, Chapter 5 gives a power series proof of LeBrun's result (originally proved by twistor methods) that all real-analytic conformal structures in three dimensions can be realised as the conformal boundary of a four-dimensional real-analytic self-dual Einstein metric. Chapters 6, 8, and 9 carefully discuss the original motivation and consequences of the ambient metric in constructing a wealth of conformal invariants (in some circumstances all scalar invariants) that would otherwise be inaccessible. In particular, Chapter 8 gives a careful discussion of jet isomorphisms that is already useful in Riemannian geometry, untangling the jets of a metric in terms of Riemannian curvature and its covariant derivatives. Chapter 7 also discusses the modifications to the construction that one can impose in the conformally flat or conformally Einstein case so as to obtain a unique expansion of all orders.

Nowadays, it seems that any of the more subtle advances in local conformal differential geometry depend on the ambient metric in an essential way. Such advances continue apace. The careful exposition provided by *The ambient metric* has been well worth the wait!

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