Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces,
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It is fair to say that for mathematicians involved in differentiating functions, modern times began with Lebesgue's proof that real-valued Lipschitz functions of a real variable are differentiable almost everywhere. This result is an early success of measure theory, and it is interesting to notice that Baire category arguments fail to provide it since the set of smooth points can be meager. Henri Lebesgue, a young researcher when this theorem was shown, scored a point in his personnal competition with René Baire through this spectacular progress.

Lebesgue's theorem is optimal as stated, since Zahorski later showed that any $G_{\delta\sigma}$ negligible subset of the real line is exactly the set where some real-valued Lipschitz function is not differentiable. However, it was not the end but rather the beginning of the story. Indeed, Lipschitzness can be defined in much greater generality for functions between metric spaces (a notion which goes back to Fréchet in the early twentieth century), and when these metric spaces are also vector spaces, it is natural to wonder whether smooth points necessarily exist, and if so whether there are many of them. This query is not over yet, but the present book, written by the three leading experts of the field, provides an authoritative and updated reference on differentiability of Lipschitz functions. It should be stressed, however, that this book is not a survey since most of it consists of new material.

The finite-dimensional theory began with Rademacher's 1919 extension of Lebesgue's theorem: every Lipschitz function between finite-dimensional normed spaces is differentiable almost everywhere with respect to the Lebesgue measure. David Preiss showed 25 years ago the existence of a negligible subset N of the plane such that the set of smooth points of any Lipschitz function from $\mathbf{R}^2$ to $\mathbf{R}$ meets N, and thus that Rademacher's theorem is not optimal. The subsequent work eventually reached its conclusion since in a very recent joint article, David Preiss and Gareth Speight showed that for any integer $n \geq 2$, the space $\mathbf{R}^n$ contains a negligible set N such that if $k < n$ and f is a Lipschitz function from $\mathbf{R}^n$ to $\mathbf{R}^k$, then the set of smooth points of f meets N. However, the main concern of the book is not the finite-dimensional theory.

Indeed the book under review is entilted *Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces*. This title clearly indicates that it focuses on infinite-dimensional theory and on Lipschitz functions between Banach spaces. In this respect, it can partly be seen as a companion to the Benyamini and Lindenstrauss book from 2000, *Geometric nonlinear functional analysis*, which is quoted throughout the text. However, the present book focuses on differentiablity issues and porosity, and the overlap between the two works is rather small.

It is well known that local compactness is a necessary condition for the existence of a Haar measure on a Polish group. Therefore, there is no obvious extension to the Rademacher theorem when the domain space is an infinite-dimensional Banach

space. Nevertheless, notions of negligibility have been introduced: Christensen's Haar-null sets, and the more restrictive class of Gauss-null sets, which coincide with Aronszajn-null sets by a theorem of Marianna Csornyei. These notions, and a number of classical concepts such as the Radon–Nikodym property, smoothness and roughness of norms, Asplund spaces, variational principles. . . are recalled in this work, making it quite self-contained, easy to read, and frequently original even into the survey part. But let us consider now the main features of the book.

First of all, the lack of compactness (and thus of uniformity) in infinite-dimensional spaces yields to two levels of smoothness for Lipschitz functions: Gâteaux-smoothness, that is, directional smoothness with linear dependence on the direction, and the usual Fréchet-smoothness where the remainder is uniform in the direction. Gâteaux-smoothness is at present quite well understood; Fréchet-smoothness remains elusive. Second, what is important in applications is not only the existence of Fréchet smooth points, but the way a function can be controlled with its behaviour at smooth points: hence the book investigates mean-value estimates and their multidimensional versions using the divergence theorem. Third, one needs a notion of a negligible set which would allow us to state that real-valued Lipschitz functions defined e.g. on the Hilbert space are "almost everywhere" differentiable. Hence the hunt for a well adapted σ-ideal of small sets is open. As explained in the introduction, this hunt is not yet fully successful. But it is already clear that differentiability questions lie so deep that they request the interplay between measure and Baire category methods, as explained below.

The Baire category theorem remains available in infinite-dimensional Banach spaces, but one cannot stay at the mere topological level to use it properly. The definition we need is a metric notion which strictly implies meagerness, namely porosity. This notion is relevant to differentiability: when a set P is porous, the function $f(x) = d(x, P)$ fails to be differentiable at every point of P. However, even in the real line, the σ-ideal generated by porous sets is smaller than the ideal of sets which are both meager and Lebesgue-null. The results of this book show that Lebesgue's theorem properly extends to spaces for which porous sets are small in sophisticated measure-theoretical ways. This sophisticated negligibilities are Γ-smallness, as introduced by Preiss and Lindenstrauss about 15 years ago, and the new notion of Γ_n-smallness (see pages 72–74 of the book): a set is Γ-null when it meets a typical (in the Baire category sense) infinite-dimensional surface, in other words a continuous directionally smooth image of the Hilbert cube, and it is Γ_n-null if it satisfies the same condition with the n-dimensional cube.

It turns out that a separable space has a separable dual exactly when its porous sets are Γ_1-null, and porous sets in the Hilbert space are Γ_2-null but in general *not* Γ_3-null (see page 169). In other words, porous sets in the Hilbert space are Lebesgue-negligible when intersected with a typical line or two-dimensional surface but not with a typical three-dimensional "space". On the other hand, porous sets in spaces whose asymptotic modulus of smoothness is smaller than any power ot t are Γ-null: this assertion applies e.g. to c_0 and illustrates the surprising fact that c_0 is somewhat smoother than l_2. It follows, with more work, that if the asymptotic modulus of smoothness of X is $o(t^n \log^{n-1}(1/t))$, then every Lipschitz function from X to $\mathbf{R}^n$ has points of Fréchet-differentiability and satisfies a multidimensional mean value estimate (see page 262). This applies in particular to the space l_n, and provides a fascinating connection between the exponent n and the dimension n.

Such results are actually optimal, and for instance every Lipschitz function from the Hilbert space to the plane $\mathbf{R}^2$ satisfies such an estimate but it is *not* so for functions with values in $\mathbf{R}^3$. Actually, it is not known whether three (or countably many) Lipschitz functions defined on the Hilbert space have a common point of Fréchet differentiability, although by the above it is so for two such functions.

This book contains much more than what is recalled here, but to keep the review under reasonable size we should rather refer the reader to the introductions to the book and to every chapter, where the contents are very clearly explained. We should be grateful to (the late) Joram Lindenstrauss, David Preiss, and Jaroslav Tišer for providing us with this splendid book which dives into the deepest fields of functional analysis, where the basic but still strange operation called differentiation is investigated. More than a century after Lebesgue, our understanding is not complete. But thanks to the contribution of these three authors, and thanks to this book, we know a fair share of beautiful theorems and challenging problems.

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