

Introduction to quantum graphs, by Gregory Berkolaiko and Peter Kuchment, *Mathematical Surveys and Monographs*, vol. 186, American Mathematical Society, Providence, RI, 2013, xiii+270 pp., US \$86.00, ISBN 978-0-8218-9211-4

The story of quantum graphs illustrates that a flower can grow only if it finds good soil. The central idea of the book under review is almost eighty years old, being first put forward by Linus Pauling as a simple model to describe certain organic molecules. It took almost two decades before the scheme was worked out in detail [RS53], and subsequently it was happily forgotten; for another third of a century it had the status of an obscure textbook example.

Looking back, this lack of interest may seem strange because the subject has various attractive features. On one hand it is simple, the building elements being ordinary differential operators. On the other hand it combines tools and intuition from a number of disciplines such as graph theory, combinatorics, mathematical physics, PDEs, and spectral theory, presenting thus nontrivial challenges which, as the authors say, make it dear to a mathematician's heart.

The theory started slowly returning to the stage in the 1980s; see [Ro83] or [GP88, ES89]. The main catalyst came from the application side, being related to the progress of fabrication techniques in solid state physics which made it possible to prepare structures of designed shapes, first from metals and semiconductors, later from other materials. Many of them consisted of microscopic “wires” on which electrons moved ballistically, and that made quantum graphs a very suitable model. What is probably more important, however, is that these new investigations attracted attention to a rich mathematical structure of the theory and triggered a wave of papers that has continued to grow. The authors of this book were a part of those efforts, the older one already at the early stages.

In a fast developing field a need naturally arises for texts which summarize the progress and indicate directions of further explorations. In the present case the first such paper was probably [KS99], followed soon by a two-part review of one of the authors [Ku04, Ku05]. A powerful impetus came from a 2007 semester-long program “Analysis on Graphs” in the Isaac Newton Institute in Cambridge. A large number of papers which originated there was collected in the proceedings volume [EKKST] and inspired many other endeavors. Important as these surveys and collections were, though, they could not substitute for a monograph-style presentation with a unique author perspective. This gap is now filled by the book under review, which, according to the authors, serves dual purposes: it provides an introduction to and survey of the current state of quantum graph theory, and at the same time, it can serve as a much needed reference text.

Even if it is not formally divided in that way, the book consists in fact of three parts differing in their style of exposition. The first two chapters describe basic constructions and frequently used technical results, the following three are devoted to various issues from the spectral theory, and the closing two chapters have more the character of a review describing connections to quantum chaos theory as well

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as various generalizations and applications. The text is accompanied by four short appendices briefly summarizing some notions from graph theory, linear operators, spectral theory, and symplectic geometry, and an extensive bibliography including more than 700 items. To make the reading smoother, most references are collected in the closing sections of each chapter, except the last one, which is to a large degree a commented survey of results.

The first chapter introduces the reader to the subject of the book describing operators on graphs. Since most mathematicians seem to be familiar with graphs, in the combinatorial sense, as a collection of vertices where the edges simply play the role of indicators of which of the vertices are connected, the authors first describe such graphs and operators on them, in the first place the (combinatorial) Laplacian, before passing to the main topic. This is useful also because such “discrete” operators are a technical tool, as described later in Section 3.6. After that metric graphs are introduced, both abstractly as simplicial complexes and illustratively as families of curves in a Euclidean space, and the required function spaces on them are defined.

This allows one to introduce second-order differential operators on such metric graphs, in the first place the Laplacian and its generalizations to Schrödinger operators, nonmagnetic as well as magnetic. The differential expressions themselves are not sufficient; to define such Hamiltonians as self-adjoint operators one has to specify conditions that match functions from the operator domain at the graph vertices. The authors provide several equivalent ways in which families of such admissible vertex conditions can be characterized, with emphasis on the formulation proposed in [Ku04] which is based on splitting the condition into Dirichlet, Neumann, and Robin parts. They also provide examples and expressions of the corresponding quadratic forms and discuss assumptions one must impose when infinite graphs are considered.

Discussion of operators on graphs continues in Chapter 2. Here scattering matrices are the departure point allowing one to write a secular equation determining the spectrum. An alternative point of view is also proposed in which one studies first-order differential operators on the graph, oriented by replacing its edges with pairs of oppositely oriented bonds. Here one considers also scattering matrices which are associated with no self-adjoint second-order operator. The question then appears when the graph Laplacian can be factorized as A^*A with a first-order operator A ; it is shown that it happens if and only if the Robin part of the vertex conditions is absent. Furthermore, the chapter contains discussion of the index theorem of differential operators on graphs, as well as their dependence on the vertex conditions. Finally, magnetic Schrödinger operators and their behavior under gauge transformations are mentioned. Some examples are worked out, usually with the “Kirchhoff” vertex coupling or its straightforward generalizations referred to usually as δ -coupling.

The next three chapters are devoted to spectra of quantum graphs. Chapter 3 contains the general theory. For compact graphs, discreteness of the spectrum is proved together with its generic simplicity, the analytic dependence on model parameters, as well as Hadamard-type formulæ and bracketing, which generalizes the usual eigenvalue interlacing results in the Sturm–Liouville theory. For infinite graphs the spectral decomposition requires also generalized eigenfunctions which are characterized here by a Schnol'-type theorem, i.e., by their subexponential growth at large distances. While quantum graphs have some properties in common with

PDEs, there are differences, the most striking probably being invalidity of the unique continuation principle. Another important topic discussed in this chapter is the “ubiquitous” Dirichlet-to-Neumann map, which here plays a role analogous to that of Weyl’s m -function for ordinary differential equations. Finally, the authors discuss the Weyl high-energy asymptotics and the trace formulæ expressing the spectral counting function in terms of closed orbits on the graph.

The topic of Chapter 4 is periodic graph systems. The corresponding Floquet–Bloch theory is developed for quantum graphs and used to analyze their band spectra. The possible existence of “flat bands”, or a point spectrum, is highlighted as a consequence of the invalidity of the unique continuation principle. Other questions addressed here are the existence of spectral gaps and the location of spectral edges; it is shown that unless the graph is essentially a one-dimensional chain, they may not correspond to the edges of the corresponding Brillouin zone.

The general discussion of spectral properties continues in Chapter 5. The topics treated here include opening of spectral gaps by “decorations” of the graph and a thorough discussion of nodal properties of the graph Hamiltonian eigenfunctions, including nodal deficiencies and their relations to the appropriate Morse indices. Also addressed here are spectral determinants as a tool of spectral analysis, together with the related ζ functions, and scattering on graphs.

As indicated above, the last two chapters have dominantly the character of a review. The sixth one is devoted to quantum graphs as model systems for studying quantum chaos, following a deep idea proposed originally in [KS97]. The aim is to analyze statistical properties of (generic) quantum graph spectra and to compare them with distributions appearing in other chaotic systems, in particular, those described by random matrix ensembles. Using the trace formulæ derived before, the authors discuss here properties of the appropriate correlation functions and indicate directions in which this analysis could be continued.

The last chapter has the most “eclectic” character, describing briefly numerous applications and generalizations of the concept of a quantum graph. The breadth of current activities makes it difficult to pursue each of the topics listed here to a considerable depth; as the authors say, it would take another book which would become obsolete by its publication date. Despite this limitation this survey is very useful covering, in particular, inverse problems on graphs (“Can one hear the shape of a quantum graph?”, etc.), other equations that can be supported by graphs (heat, wave, Dirac, pseudo-differential operators, nonlinear Schrödinger, and others), graph models of thin network structures (with implications for interpretation of the vertex coupling conditions), “leaky” graph structures and photonic crystals, as well as the use of quantum-graph methods to model various physical phenomena.

As is clear from this description, the style of the book inevitably changes as the reader proceeds towards the end. The opening part presents theorems with complete proofs, later on some are offered in a sketched form, and the survey part contains claims accompanied with a discussion, sometimes on a heuristic level. What is important, however, is that each claim is presented with its full background in the literature, which allows the interested reader to follow the idea to the extent of what is presently known about the particular topic.

The authors say that they intended to write the book in a way accessible to graduate and advanced undergraduate students in mathematics, physics and engineering. This goal was no doubt achieved, but in addition the book can be useful to a broad spectrum of researchers interested in mathematics of quantum graphs as

well as in applications of the theory. In addition, with a reasonable degree of certainty one can predict that by filling an important gap in the literature, Berkolaiko and Kuchment have stimulated further development in this area.

REFERENCES

- [EKKST] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, A. Teplyaev, Editors, *Analysis on graphs and its applications*, Papers from the program held in Cambridge, January 8–June 29, 2007, Proc. Symp. Pure Math., vol. 77; Amer. Math. Soc., Providence, R.I., 2008. MR2459860 (2009i:00008)
- [EŠ89] P. Exner and P. Šeba, *Free quantum motion on a branching graph*, Rep. Math. Phys. **28** (1989), no. 1, 7–26, DOI 10.1016/0034-4877(89)90023-2. MR1109248 (92e:81019)
- [KS99] V. Kostykin and R. Schrader, *Kirchhoff's rule for quantum wires*, J. Phys. A **32** (1999), no. 4, 595–630, DOI 10.1088/0305-4470/32/4/006. MR1671833 (99m:81280)
- [KS97] T. Kottos and U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett. **79** (1997), 4794–4797.
- [Ku04] Peter Kuchment, *Quantum graphs. I. Some basic structures*, Waves Random Media **14** (2004), no. 1, S107–S128, DOI 10.1088/0959-7174/14/1/014. Special section on quantum graphs. MR2042548 (2005h:81148)
- [Ku05] Peter Kuchment, *Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A **38** (2005), no. 22, 4887–4900, DOI 10.1088/0305-4470/38/22/013. MR2148631 (2006a:81035)
- [GP88] N. I. Gerasimenko and B. S. Pavlov, *A scattering problem on noncompact graphs*, Teoret. Mat. Fiz. **74** (1988), no. 3, 345–359, DOI 10.1007/BF01016616 (Russian, with English summary); English transl., Theoret. and Math. Phys. **74** (1988), no. 3, 230–240. MR953298 (90f:47010)
- [Ro83] Jean-Pierre Roth, *Spectre du laplacien sur un graphe*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 19, 793–795, DOI 10.1007/BFb0100128 (French, with English summary). MR711833 (84g:58104)
- [RS53] K. Ruedenberg and C. W. Scherr, *Free-electron network model for conjugated systems, I. Theory*, J. Chem. Phys. **21** (1953), 1565–1581.

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