BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 3, July 2014, Pages 519–526
S 0273-0979(2014)01452-1
Article electronically published on March 11, 2014

Exterior billiards: systems with impacts outside bounded domains, by A. Plakhov, Springer, New York, 2012, xiii+284 pp., ISBN 978-1-4614-4480-0, US $109.00

Proposition XXXIV, Theorem XXVIII of Newton's *Principia* reads:

> If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be but half as great as that of the cylinder

(quoting from [9]). Newton gives a page and a half long proof of this result, and continues with *Scholium*,[1] also a little more than a page long, in which one of the oldest problems of calculus of variations[2] is described, with a complete answer, but without proof (Newton says, "By the same method…").

The problem is to find the shape of the body of revolution with a fixed cross-section and height having the smallest resistance to the motion in a homogeneous inviscid and incompressible medium. Specifically, the assumption made about the medium is that it consists of point masses that

(1) do not interact with each other, and
(2) reflect off the moving body elastically, so that the energy and momentum are preserved.

Newton starts with analyzing a particular case, the frustum of a cone, and presents the answer geometrically: if Q is the midpoint of the segment OD, then the optimal shape is characterized by the equality $QC = QS$; see Figure 1 on the left (the figure is taken from [9], the body is moving to the right). Then Newton considers a general case, see Figure 1 on the right. The optimal shape has a flat front end with the angle FGB equal to 135°. Newton says,

> This Proposition I conceive may be of use in the building of ships.[3]

Again, the answer is presented geometrically. Let N be a general point of the optimal profile, and let GR be parallel to the tangent line to the curve at N. Then

$$\frac{MN}{GR} = \frac{GR^3}{4BR \cdot GB^2}.$$

See Figure 2, borrowed from [4], for computer drawings of optimal shapes, and [4–6, 15] for the history of Newton's problem of minimal resistance.

The book under review is a modern continuation of Newton's work. One can imagine numerous variations on the above-described setup: for example, the moving body may lack rotational symmetry, fail to be convex, rotate while moving, have a rough surface, and so on and so forth. The assumptions (1) and (2) are still in effect, and this relates the material with geometric optics and the theory of mathematical

---

2010 *Mathematics Subject Classification.* Primary 37D50, 49Q10, 78A05.
Supported by the NSF grant DMS-1105442.

[1] explanatory comment

[2] preceding the famous *Brachistochrone*

[3] Newton's solution was criticized as unrealistic. Indeed, his model poorly describes the motion of a ship in water but it gives good results for bodies that move in rarefied gas with high Mach number. Thus Newton was ahead of his time by some 300 years!

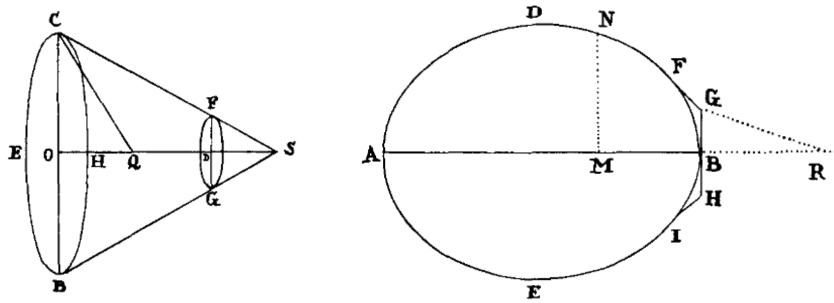

FIGURE 1

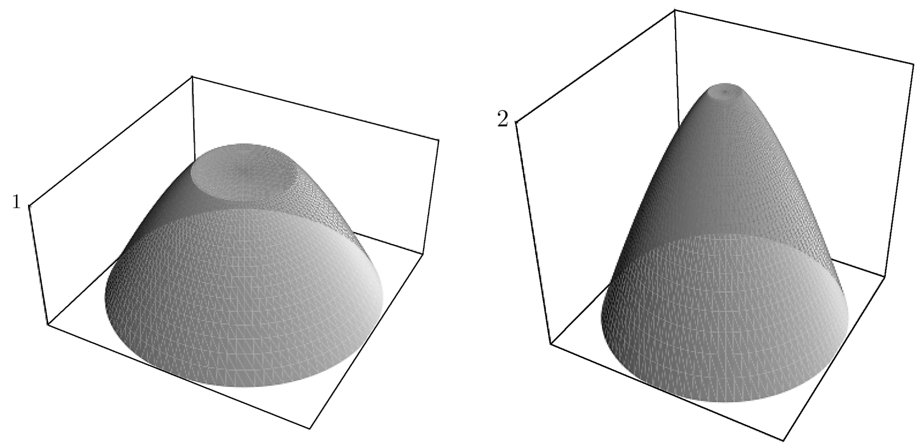

FIGURE 2. Two optimal shapes: the height equals the radius, and the height equals twice the radius.

billiards. Indeed, one may assume that the body is at rest, whereas the particles move and reflect off the surface of the body according to the law "the angle of incidence equals the angle of reflection". This connection with billiards explains the title of the book.

The book can be read at three levels: one can get a snapshot of the theory from a well-written Introduction (four pages long); one can read the 20-page-long Chapter 1 which gives a panorama of the main definitions and results, without proofs; or one can immerse into the study of the book as a whole. The book ends with a list of open problems. A substantial part of the work was done, over the years, by the author and his collaborators. I shall present a few examples of the problems tackled in the book.

Allowing not necessarily rotationally symmetric, but still convex, bodies yields shapes with smaller resistance than those of Newton's; see Figure 3, also borrowed from [4]. However, if one allows nonconvex shapes (so that a particle may reflect more than once off the surface), the result is somewhat surprising: one can construct bodies, inscribed into a given cylinder, with arbitrarily small resistance. Furthermore, this can be achieved by making arbitrarily small grooves on the surface of a convex body.

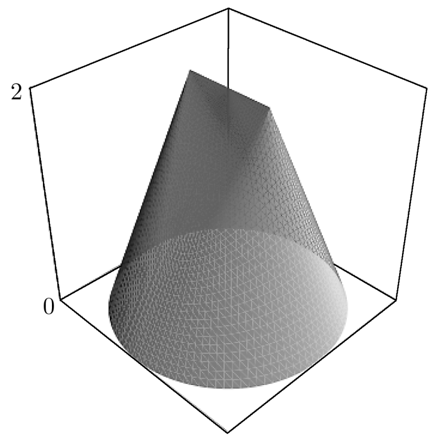

FIGURE 3. This body, discovered by P. Guasoni in the 1990s, has the same height to radius ratio but a smaller resistance than the body in Figure 2, on the right.

A significant part of the book concerns rough bodies.[4] Imagine a convex body whose surface is poked with microscopic hollows, grooves, and cracks. Such a body will seem convex but the billiard reflection from its surface will not be subject to the law "the angle of incidence equals the angle of reflection". One defines a rough body as a limit of a sequence of bodies with hollows of sizes tending to zero such that the sequence of the respective scattering laws converges in an appropriate sense. As a result, one describes this billiard scattering by a certain measure on the set of incoming and outgoing directions.

To be specific, consider the two-dimensional case. The incoming and outgoing rays are characterized by the angles φ and φ_+ made by the rays with the normal to the boundary at the impact point. The scattering law of a rough body is a measure on $[-\pi/2, \pi/2]^2$. One of the results is that this measure can be weakly approximated by scattering laws of a sequence of converging bodies with hollows if and only if it satisfies two conditions: it is invariant under the time-reversing involution $(\varphi, \varphi_+) \mapsto (\varphi_+, \varphi)$, and its two projections on $[-\pi/2, \pi/2]$ coincide with the usual billiard measure $(1/2)\cos\varphi\, d\varphi$. Similar characterizations are given in higher dimensions.

The study of rough bodies naturally lead to the Monge–Kantorovich optimal mass transport problem [16]. The general setup is as follows. One has two measure spaces (X_1, μ_1) and (X_2, μ_2) with $\mu_1(X_1) = \mu_2(X_2)$ and a cost function $c : X_1 \times X_2 \to \mathbb{R}$. One considers the set of measures Γ on $X_1 \times X_2$ whose projections on X_1 and X_2 are μ_1 and μ_2. The problem is to minimize

$$\int_{X_1 \times X_2} c(x, y)\, d\nu(x, y)$$

over $\nu \in \Gamma$.

[4] There is a significant recent activity on drag reduction in aerodynamics; see, e.g., a thematic issue [17]. The model used in this approach is totally different from the one in the book under review.

One has the following interpretation in terms of billiard scattering by rough surfaces. The sets of incoming and outgoing trajectories at a point are identified with unit hemispheres (the spaces X_1 and X_2); the measures μ_1 and μ_2 describe the densities of the trajectories. The measure ν describes the scattering at the point, and the cost function is the momentum transmitted to the body by the particle with the corresponding incoming and outgoing velocities. From this point of view, the transport corresponding to reflections off a smooth surface (the optical reflection) is not optimal: one can reduce the total cost by introducing roughness.

Another problem studied in the book concerns the mean resistance of a body. In this formulation, the body moves with the velocity randomly chosen from the uniform distribution on the unit sphere, and the resistance is considered as a random variable. One wants to minimize and/or maximize the mathematical expectation. A physical interpretation of this setting is that the body moves translationally and slowly rotates at the same time.

There are a number of subproblems of this kind, depending on the class of bodies under consideration. For example, in dimension two, the ratio of the least mean resistance in the class of nonconvex bodies of fixed area to the least mean resistance in the class of convex bodies of the same area is about 0.9878 (for convex bodies, the optimal shape is a round disk). Another example: given a convex body, by how much could its mean resistance be lowered by grooving its surface? The answer does not depend on the body, but only on the dimension d. For $d = 3$, the answer is about 3%, and as $d \to \infty$, the answer tends to approximately 20.9%.

The Magnus effect is a deflection of the trajectory of a spinning body (such as a golf or a tennis ball). Named after a German physicist who described it in the mid-nineteenth century, it was described by Newton in [10]:[5]

> ...I had often seen a Tennis ball, struck with an oblique Racket, describe such a curve line. For, a circular as well as a progressive motion being communicated to it by that stroak, its parts on that side, where the motions conspire, must press and beat the contiguous Air more violently than on the other, and there excite a reluctancy and reaction of the Air proportionably greater.

One of the topics of the book is a novel model of the Magnus effect where the reflections of the particles from the body are elastic but the surface is rough (in the usual models, the interaction of the particles with the body is not elastic). This study is restricted to the case of a two-dimensional rough disk.

The Magnus effect is direct if the moving rough disk is deflected in the direction of rotation of its front points, and inverse otherwise. An important quantity is $\gamma = \omega r/v$, where ω is the angular velocity of the rotation, r is the radius, and v the speed of the disk. One finds a general formula for the resistance force and its moment in terms of the scattering law of the rough disk, along with theoretical and numerical results for particular profiles of hollows.

For example, consider the sequence of regular m-gons with m congruent rectangles with aspect ratio $1 : m$ removed. The smaller side of each rectangle is contained in a side of the m-gon, and the ratio of the longer side of a rectangle to the side of the polygon equals $1 - 1/mp$; see Figure 4 (a). This sequence represents a rough disk with rectangular hollows. Its moment of resistance force equals -0.75γ, hence one has an inverse Magnus effect in this case. For the rough disks with triangular

[5]One cannot help mentioning Berry's law: "Nothing is ever discovered for the first time" [2].

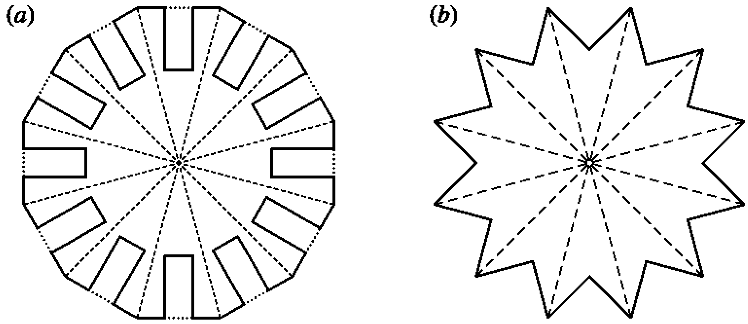

FIGURE 4. Two kinds of rough disks; the figure borrowed from [13].

hollows, Figure 4 (b), the moment of the resistance is known only in some particular cases. Incidentally, although it is known that some rough disks exhibit the direct Magnus effect, explicit examples of such roughness are not known.

The last two chapters of the book concern invisibility and retroreflection. Invisibility is a subject of a great current interest [7, 8]. In the book under review, this and related terms are understood in the sense of geometric optics. In fact, three related properties are considered:

(1) a body has zero resistance in a given direction;
(2) a body leaves no trace when moving in a given direction;
(3) the body is invisible in a given direction.

The relation between these properties is (3) $\subset$ (2) $\subset$ (1).

To be concrete, let us define (3): a body B is invisible in direction v if every incoming ray of light having direction v continues along the same line after its reflections off B. There is a wealth of results on invisible bodies; here is a sampler.

First of all, bodies invisible in one direction exist; see Figure 5. The curves are parabolas sharing the focus. The body of revolution of this plane figure is invisible in dimension three.

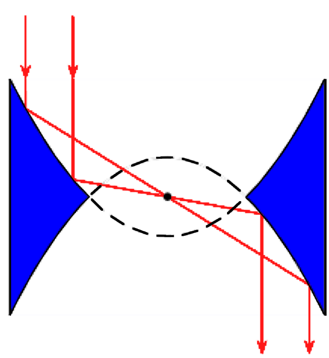

FIGURE 5. An invisible body; the figure borrowed from [12].

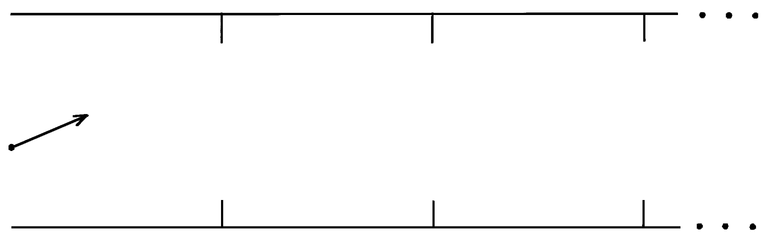

Figure 6. A tube; the width is unit, and the obstacles have height ε. The figure borrowed from [1].

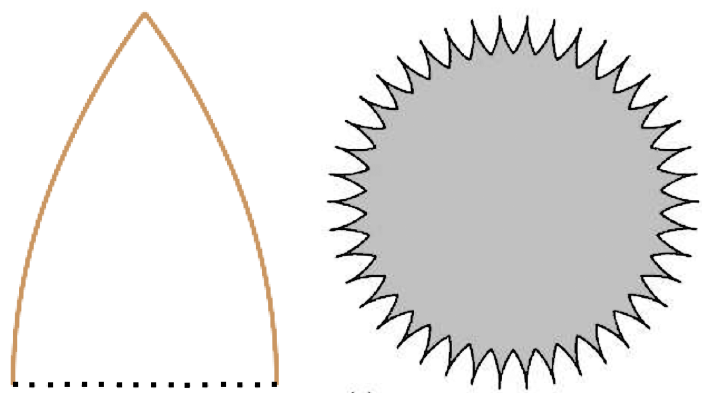

Figure 7. The helmet and a retroreflector based on it. The figure borrowed from [11].

One can construct three-dimensional bodies invisible in two directions, or invisible from a point. However bodies invisible in all directions do not exist, and it is not known whether bodies invisible in three directions exist. It is worth mentioning a recent related result of Bialy [3] that there exists a smooth Riemannian metric in $\mathbb{R}^n$ that is Euclidean outside a compact set and that has $n(n+1)/2$ invisible directions.

A retroreflector is a reflecting body such that every incoming ray of light eventually reflects in the opposite direction. Everyone is familiar with the retroreflector made of three flat pairwise orthogonal mirrors (bicycle cube reflector). However this reflector is imperfect: if it has a finite size, then some rays do not reflect in the opposite direction. No perfect retroreflector is known, and the last chapter of the book concerns asymptotically perfect retroreflectors, that is, sequences of bodies whose reflection properties approximate perfect retroreflection.

Namely, four examples of asymptotically perfect retroreflectors are considered, and here we present two. The first is a semi-infinite tube depicted in Figure 6. The result is that, for every $\varepsilon < 1/2$, almost every trajectory leaves the tube, and as $\varepsilon \to 0$, the measure of the set of incoming trajectories that exit in the opposite direction tends to 1.

Another remarkable hollow, discovered by P. Gouveia and called helmet, is made of two arcs of parabolas that share the axis; the focus of each parabola lies on the other parabola; see Figure 7. The helmet is a nearly perfect retroreflector, with an error of less than 1%.

Let me conclude with a remark. Geometric optics and the theory of mathematical billiards are intimately related with symplectic geometry: the space of oriented nonparameterized geodesics (rays of light) is a symplectic manifold, and the optical (billiard) reflection is a symplectic transformation; see, e.g., [14]. This symplectic point of view may bring new insights into the area (in fact, [3] makes use of symplectic geometry in a substantial way).

To conclude, this is a fascinating book, well written and well illustrated, and, most importantly, open-ended. I expect substantial progress to be made in this area in the near future.

References

[1] Pavel Bachurin, Konstantin Khanin, Jens Marklof, and Alexander Plakhov, *Perfect retroreflectors and billiard dynamics*, J. Mod. Dyn. **5** (2011), no. 1, 33–48, DOI 10.3934/jmd.2011.5.33. MR2787596 (2012d:37019)

[2] M. Berry, *Three laws of discovery in* `http://michaelberryphysics.wordpress.com/quotations/`

[3] M. Bialy, *A remark on the number of invisible directions for a smooth Riemannian metric*, arXiv:1305.2736.

[4] Giuseppe Buttazzo, *A survey on the Newton problem of optimal profiles*, Variational analysis and aerospace engineering, Springer Optim. Appl., vol. 33, Springer, New York, 2009, pp. 33–48, DOI 10.1007/978-0-387-95857-6_3. MR2573703

[5] Giuseppe Buttazzo and Bernhard Kawohl, *On Newton's problem of minimal resistance*, Math. Intelligencer **15** (1993), no. 4, 7–12, DOI 10.1007/BF03024318. MR1240664

[6] Herman H. Goldstine, *A history of the calculus of variations from the 17th through the 19th century*, Studies in the History of Mathematics and Physical Sciences, vol. 5, Springer-Verlag, New York, 1980. MR601774 (83i:01036)

[7] Allan Greenleaf, Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann, *Invisibility and inverse problems*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 1, 55–97, DOI 10.1090/S0273-0979-08-01232-9. MR2457072 (2010d:35399)

[8] Ulf Leonhardt and Thomas Philbin, *Geometry and light. The science of invisibility*, Dover Publications Inc., Mineola, NY, 2010. MR2798945 (2012f:53001)

[9] Isaac Newton, *Mathematical principles of natural philosophy. Vol. 1: The motion of bodies*, Translated into English by Andrew Motte in 1729. The translations revised, and supplied with an historical and explanatory appendix, by Florian Cajori, University of California Press, Berkeley, 1962. MR0175747 (31 #24a)

[10] I. Newton, *A letter of Mr. Isaac Newton, of the University of Cambridge, containing his new theory about light and color*, Philosophical Transactions of the Royal Society **6** (1671), 3075–3087.

[11] Alexander Plakhov, *Mathematical retroreflectors*, Discrete Contin. Dyn. Syst. **30** (2011), no. 4, 1211–1235, DOI 10.3934/dcds.2011.30.1211. MR2812962 (2012f:37080)

[12] Alexander Plakhov and Vera Roshchina, *Invisibility in billiards*, Nonlinearity **24** (2011), no. 3, 847–854, DOI 10.1088/0951-7715/24/3/007. MR2772626 (2012a:37072)

[13] Alexander Plakhov, Tatiana Tchemisova, and Paulo Gouveia, *Spinning rough disc moving in a rarefied medium*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **466** (2010), no. 2119, 2033–2055, DOI 10.1098/rspa.2009.0518. MR2652732 (2011j:76152)

[14] Serge Tabachnikov, *Billiards* (English, with English and French summaries), Panor. Synth. **1** (1995), vi+142. MR1328336 (96c:58134)

[15] V. M. Tikhomirov, *Stories about maxima and minima*, Mathematical World, vol. 1, American Mathematical Society, Providence, RI, 1990. Translated from the 1986 Russian original by Abe Shenitzer. MR1152027 (92k:49002)

[16] Cédric Villani, *Optimal transport: old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. MR2459454 (2010f:49001)

[17] *Theme Issue "Flow-control approaches to drag reduction in aerodynamics: progress and prospects"*, Phil. Trans. of the Royal Soc. A, **369**, 2011.

Serge Tabachnikov
Department of Mathematics
The Pennsylvania State University
University Park, Pennsylvania
E-mail address: tabachni@math.psu.edu