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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
DRAGOVIĆ AND RADNOVIĆ

MR2798784 (2012j:14059) 14J70; 37D50, 37J35

Dragović, Vladimir; Radnović, Milena

Poncelet porisms and beyond.

Integrable billiards, hyperelliptic Jacobians and pencils of quartics.

Frontiers in Mathematics.

Birkhauser/Springer Basel AG, *Basel*, 2011, viii+293 pp., $74.95,
ISBN 978-3-0348-0014-3

Jean-Victor Poncelet was among the classical French mathematicians who worked in mechanical engineering; the famous theorem that now bears his name was part of the "Saratov notebook", the notes on geometry he wrote during his yearlong imprisonment after the battle of Krasnoi, where he served as Lieutenant in Napoleon's army. From the many fascinating historical accounts (for mathematical perspective cf., e.g., [Rend. Sem. Mat. Fis. Milano **54** (1984), 145–158 (1987); MR0909049 (88j:14064)] by H. J. M. Bos as well as its review by S. L. Kleiman) we learn that in order to give a rigorous proof of his theorem he had the vision to extend Euclidean geometry by two kinds of invisible points, the points at infinity of projective geometry and the complex points which are not defined over the reals. In the online version (`http://www.crcnetbase.com/isbn/978-1-58488-347-0`) of E. W. Weisstein's *CRC concise encyclopedia of mathematics* [second edition, Chapman & Hall/CRC, Boca Raton, FL, 2003; MR1944431 (2003j:00008)] we can watch the theorem in motion: it says that the billiard in a conic is a completely integrable system. It is perhaps beside the point to seek a precise attribution for extending this result to the billiard in a quadric hypersurface, but the work of S.-J. Chang and R. M. Friedberg [J. Math. Phys. **29** (1988), no. 7, 1537–1550; MR0946326 (89j:58043)] was pioneering in that respect; a sequel [S.-J. Chang, B. Crespi and K. J. Shi, J. Math. Phys. **34** (1993), no. 6, 2242–2256; MR1218986 (94g:58092)] related it both to Darboux's theorem and to the "space Poncelet" theorem by P. A. Griffiths and J. D. Harris [Comment. Math. Helv. **52** (1977), no. 2, 145–160; MR0498606 (58 #16695)]; and in [J. Math. Phys. **34** (1993), no. 6, 2257–2289; MR1218987 (94g:58093)] Crespi, Chang and Shi gave explicit genus-2 solutions to the billiard problem in space, using Klein's generalization to genus 2 of the Weierstrass σ-function, which at that time was little known but twenty years later has become one of the major tools for studying PDEs and moduli spaces.

In any event, the integrability of the billiard in a quadric hypersurface and the expression of the trajectories by hyperelliptic functions are the main themes of this book. The authors (Radnović first worked on Poncelet's theorem in her Ph.D. thesis [*Geometrija integrabilnih bilijara i periodične trajektorije*, Univ. Belgrade, 2003]; Dragović was the advisor) have written a great many papers that reveal connections of the theorem with other classical results and in particular Cayley's algebraic closure condition (see the next paragraph of this review), which could be relevant to coding theory and cryptography. This book is mainly devoted to the authors' work, richly embedded in its historical perspective and adapted to the latest implications in the theory of integrable systems and PDEs, including statistical mechanics. There is no aim at completeness in any respect, be it bibliographical, historical, algebraic/geometric or dynamical; this shows excellent restraint, and it makes the book very valuable in its selective purpose. A book attempting to address all the meanings and implications of Poncelet's theorem by now would have to have the nature of a handbook. To put it in context, I avail myself of a second-hand quote, namely Nicholas Katz' paraphrase of Lipman Bers' assertion that "[a] significant mathematical problem [...] is never solved only once" [N. M. Katz, in *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974)*, 537–557, Amer. Math. Soc., Providence, RI, 1976; MR0432640 (55 #5627) (p. 542); L. Bers, ibid., 559–609; MR0427623 (55 #654) (p. 559)]: Poncelet's theorem has been given essentially different proofs by enumerative geometry, synthetic geometry, algebraic geometry, analysis, mechanics, measure theory and matrix algebra, at least.

The authors' point of departure is the proof of Poncelet's theorem given (in its algebro-geometric version most conceptually, if not for the first time) by Griffiths and Harris [op. cit.] as well as M. Reid's, P. E. Newstead's and R. Donagi's expression of the addition law on a hyperelliptic Jacobian by means of linear subspaces of the intersection of two quadric hypersurfaces. Roughly speaking, a pencil of quadrics in $(2g+1)$-dimensional projective space corresponds to a hyperelliptic curve of genus g (an idea that goes back to A. Weil [cf. Amer. J. Math. **76** (1954), 347–350; MR0061125 (15,778c)]); the Poncelet case of plane conics (genus 1) takes only a little tweaking to be represented in $\mathbb{P}^3$, where it was unwittingly rediscovered several times in the 19th century. The authors provide a "just-in-time" review of the analytic and geometric version of the elliptic and hyperelliptic addition rule; projective geometry and duality, including some beautiful (and to-the-point) classical theorems with proofs; and correspondences. They devote one chapter each to Poncelet-type theorems which can be translated into a hyperelliptic addition law. The more traditional billiard in a quadric was interpreted as a Poncelet-type result by A. P. Veselov [Funktsional. Anal. i Prilozhen. **22** (1988), no. 2, 1–13, 96; MR0947601 (90a:58081)], who also did further work with J. Moser on discrete geodesic motion. Cayley's resultant-type condition for the Poncelet closure is explored; this could be applied to coding theory and cryptography, being an algebraic condition on periodic orbits. Then, the authors offer a link between Poncelet's theorem and Marden's (a mid-20th-century result on the relationship between the roots of a cubic polynomial and those of its derivative). In developing continued-fraction representations occurring in Marden's theorem, a relation with the integrals of the Toda system is pointed out. A chapter on the theme of continued fractions revisits the link found by Halphen between the value of an elliptic integral and Poncelet's theorem. The last chapter is the most contemporary. Elliptic functions were used by R. J. Baxter to identify solvable models in statistical mechanics; as in the case of the soliton equations, the theoretical underpinnings gave rise to interpretations and developments in representation theory and a host of related subjects. The authors encode some of the solutions, classified by I. M. Krichever, by means of a Poncelet $(2,2)$ correspondence: on the one hand, Baxter's biquadratic; on the other, points of the incidence correspondence that pairs points on a conic and tangents to another conic.

The book provides a self-guided introduction to a set of classical gems that have to do with elliptic or hyperelliptic addition theorems; several connections were worked out originally by the authors in previous research papers. In addition, solutions to some of the integrable systems that came of age in the 1970s (KdV hierarchy, Toda lattice, Yang-Baxter equations, chiefly) are produced in this context. The style is clear and the calculations are complete. The authors should be commended for refraining from trying to survey more directions into which the Poncelet pied piper could take the charmed reader; had they not, we would be reviewing an infinite book. This reviewer is not gifted with such tasteful restraint, but will only offer a minimal complement—the references provided above also are not in this book. On the classical side, one can find a wealth of historical perspectives and related results in H. F. Baker's treatment of the Poncelet theorem [*Principles of geometry. Vol. 4*, Cambridge Univ. Press, Cambridge, 1925; JFM 51.0531.07]. Of other related theorems on linear systems on a plane curve (not necessarily of genus 0 or 1 as in the original), the present book pursues Darboux's but not C. Segre's [Torino Atti **59** (1924), 303–320; JFM 50.0428.02]—credit is due to Ciro Ciliberto for this little-known reference. Peter F. Ash provided the charming reference to Chapter 14 of I. J. Schoenberg's [*Mathematical time exposures*, Math. Assoc. America, Washington, DC, 1982; MR0711022 (85b:00001)], which shows in particular that Steiner's theorem is a singular case of Poncelet—which is a good excuse for having fun with the conformal-mapping software developed by D. T. Piele, M. W. Firebaugh and R. Manulik [Amer. Math. Monthly **84** (1977), no. 9, 677–692; MR0486568 (58 #6291)]. Of course Poncelet's theorem never ceases to rise from its ashes; two reincarnations will have to suffice: hyperelliptic curves "with real multiplication" (a case investigated by G. Humbert in 1899) [cf. J. M. Wilson, Acta Arith. **93** (2000), no. 2, 121–138; MR1757185 (2001f:11099)]; and an occurrence in the theory of Painlevé transcendents [N. J. Hitchin, in *Geometry and analysis (Bombay, 1992)*, 151–185, Tata Inst. Fund. Res., Bombay, 1995; MR1351506 (97d:32042)]. This is also not the first monograph exclusively devoted to Poncelet's theorem [cf., e.g., L. Flatto, *Poncelet's theorem*, Amer. Math. Soc., Providence, RI, 2009; MR2465164 (2011f:37001)], but it is certainly a focused, enjoyable and valuable one.

Emma Previato
From MathSciNet, March 2014

MR2465164 (2011f:37001) 37-01; 14H52, 14H70, 35E05, 37D40, 37D50, 51-01, 51A05

Flatto, Leopold

Poncelet's theorem.

(Chapter 15 by S. Tabachnikov)
American Mathematical Society, *Providence, RI*, 2009, xvi+240 pp.,
ISBN 978-0-8218-4375-8

What do shooting pool [J. K. Moser and A. P. Veselov, Comm. Math. Phys. **139** (1991), no. 2, 217–243; MR1120138 (92g:58054)]; two-demand queueing probabilities; instantons [M. S. Narasimhan and G. Trautmann, Pacific J. Math. **145** (1990), no. 2, 255–365; MR1069891 (91m:14016)]; the convex hull of the eigenvalues of a complex matrix (more precisely, Toeplitz' numerical range [cf. B. Mirman, Linear Algebra Appl. **281** (1998), no. 1-3, 59–85; MR1645335 (99j:51013)]); the twist map on an annulus [E. Garibaldi and A. O. Lopes, Ergodic Theory Dynam. Systems **28** (2008), no. 3, 791–815; MR2422016 (2010b:37059)]; the Dirichlet problem in a planar domain [V. P. Burskii and A. S. Zhedanov, "On Dirichlet, Poncelet and Abel problems", preprint, `arxiv.org/abs/0903.2531`]; and genus-2 curves with real multiplication [J.-F. Mestre, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 13, 721–724; MR0972820 (89k:11106)] have in common? "One of the most important ... theorems in projective geometry", announces the book under review (Preface), namely Poncelet's. I, very respectfully, disagree. The true "important" feature that has those and many more manifestations is the addition law for an elliptic curve (cf. [P. A. Griffiths, Invent. Math. **35** (1976), 321–390; MR0435074 (55 #8036)], where the theorem was derived from this fact). The double nature, analytic and algebraic, of, more generally, Riemann surfaces is perhaps a small part of the reason for "the unreasonable effectiveness of mathematics in the natural sciences" [E. P. Wigner, in *Mathematical analysis of physical systems*, 1–14, Van Nostrand Reinhold, New York, 1985; MR0824292]. Motion, and its organization under a group law, is—as far as we currently know—the stuff that nature is made of. The fascination of Poncelet's theorem, to me, rests on the fact that it can be proved in so many fundamentally different ways. Poncelet himself (while a prisoner in Russia, as his romantic biography at the St. Andrews' history site informs) proved the theorem by synthetic geometry, bypassing the fact that the complex projective plane had not been formally defined; a visionary feat. In that spirit, the theorem is a statement about two ellipses, and intersecting or tangent lines, a "porism" which says that if a polygon of n sides is inscribed in an ellipse and circumscribed to another, then infinitely many such polygons exist (in fact, the given one can be rotated in a suitable sense). This is very well, but why does the inner ellipse happen to be *the* ellipse enveloped by the n-sided polygons *of maximal perimeter* inscribed in the outer ellipse [G. Fejes Tóth and L. Fejes Tóth, Period. Math. Hungar. **3** (1973), no. 3-4, 271–274; MR0333990 (48 #12309)]? (Parenthetically, dynamics scholars now recognize the inner ellipse as the caustic of a billiard system.) My favorite explanation of course is that zig-zagging along the sides of the polygon is a measure-invariant motion, and the measure, as Jacobi pointed out, is the arc length of the (given, outer) ellipse: an elliptic integral; moving along the sides then is an integrable system (continuous or discrete, as you may prefer) and amounts to the addition of one given point (of period n, in this case) on the corresponding elliptic curve. Then again, there are independent proofs given by enumerative geometry and more.

But that is not the point of view taken in this book. Instead, the author aims for a "textbook", resulting as it does from his sabbatical year at the NSA, where the course was taught. It has then the great merit of being reasonably self-contained: a sketch of projective geometry is given, and in view of proving the theorem using the elliptic curve, an introduction to complex analysis (conformal mappings, doubly-periodic meromorphic functions, modular functions); the algebraic transliteration of the n-closure given by Cayley; an array of pretty "degenerate cases" (the two ellipses are in special position: another plethora of classical theorems ensues); the queueing-theory application (this is part of the author's research area). As a self-study book, it seems a little challenging for a newcomer to complex analysis (an excellent and down-to-earth alternative introduction to elliptic functions and modular forms could be P. Du Val's [*Elliptic functions and elliptic curves*, Cambridge Univ. Press, London, 1973; MR0379512 (52 #417)]), and it's a pity there are no exercises. Minor quibbles (such as the fact that $\widehat{\mathbb{C}}$ is called "the complex sphere", which might be a bit ambiguous, instead of "the Riemann sphere"; quite a few typos; not many pictures) are best suppressed, in view of the fact that the book is certainly well written. The real regret I have, quite aside from the fact that the other aspects of the theorem are not considered, is the fact that there is no mention of the extension of the theorem in space (hyperelliptic curves), except in the very beautiful Supplement on billiards (Chapter 15), contributed by S. Tabachnikov. However, I fully agree with the author: Poncelet's theorem is "one of the most ... beautiful theorems in projective geometry"; truly, it is beautiful, and so is this book.

Emma Previato

From MathSciNet, March 2014

MR0498606; (58 #16695) 14J99; 14D20, 14N05

Griffiths, Phillip; Harris, Joe

A Poncelet theorem in space.

Commentarii Mathematici Helvetici **52** (1977), *no.* 2, 145–160.

The classical theorem of Poncelet on polygons inscribed in a curve of second order and circumscribed around another such curve is generalized to the case of three-dimensional space in the following way. Let S and S' be smooth surfaces of second order in the complex projective space $\mathbf{P}^3$. Then the existence of a finite polyhedron simultaneously inscribed in and circumscribed around the surfaces S and S' implies the existence of infinitely many such polyhedra. The vertices of these polyhedra lie on the intersection curve of the surfaces S and S', their faces are tangent to both surfaces, and the edges are lines lying alternately on S and S'. The paper is presented in a rather elementary way; the proofs of the theorems contained in it can be understood even by a nonexpert in algebraic geometry.

M. A. Akivis

From MathSciNet, March 2014

MR0909049 (88j:14064) 14N05; 01A55, 14-03

Bos, H. J. M.

The closure theorem of Poncelet.

Rendiconti del Seminario Matematico e Fisico di Milano **54** (1984), 145–158 (1987).

This note summarizes some joint work done with F. Oort, C. Kers, and D. Raven on the history and mathematics of Poncelet's famous closure theorem, which today is formulated as follows: given two smooth conics C, D in general position in the complex projective plane (in fact, although this is not stated, any algebraically closed ground field of characteristic not 2 will do), if there exists one nondegenerate n-gon inscribed in C and circumscribed about D, then there exist infinitely many, or equivalently, if there exists one nondegenerate sequence of pairs $(P_1, L_1), (P_2, L_2), \dots$ with $P_i \in C_i$ and L_i tangent to D_i and $P_i, P_{i+1} \in L_i$ such that $P_{n+1} = P_1$, then there exist infinitely many. There are always four degenerate n-gons, respectively sequences: if $n = 2j$, they arise from the four choices of $P_j \in C \cap D$; if $n = 2j + 1$, they arise from the four choices of L_j as a common tangent. Three proofs are discussed, Poncelet's (1822), Jacobi's (1828), and Griffiths' (1976), but not A. Hurwitz's [Math. Ann. **15** (1879), 8-16; Jbuch **9**, 398]. Hurwitz in effect considered all possible sequences and formed the locus G of (P_i, P_{n+1}) in $C \times C$. If P_1 is a general point, then there are two distinct possibilities for P_{n+1}; so $\int [G] \cdot [C \times P_{n+1}] = 2$. Similarly $\int [G] \cdot [P_{n+1} \times C] = 2$. Hence $\int [G] \cdot [\Delta] = 4$, where $[\Delta]$ is the diagonal. However, G meets Δ in at least five distinct points. Therefore G contains Δ. Griffiths considered the locus E of (P_1, L_1). It is a double covering of C, branched over $C \cap D$. So E is an elliptic curve. Since the automorphism of E that sends (P_1, L_1) to (P_{n+1}, L_{n+1}) has five fixed points, it is constant. Jacobi used his theory of elliptic curves and argued similarly, but the degree of similarity is a delicate matter, the stuff of historical analysis. Poncelet's "brilliant, complicated and idiosyncratic" proof centers on essentially the following lemma: if $C, D_1, D_2, \cdots$ are conics through the same four distinct points and if $(P_1, L_1), (P_2, L_2), \cdots$ is a sequence as above but with L_i tangent to D_i instead of D, then as P_1 varies continuously along C, the chord P_iP_n envelops (is tangent to) another conic D' through the four points. The theorem follows by taking $D_i = D$ for $i = 1, \cdots, n$; indeed, the conics D and D' have five tangents in common, so they coincide. At this point, it is incorrectly asserted that $D = D'$ because they have four points and three tangents in common; however, any two conics have four points and four tangents in common. The lemma is reduced to the case $n = 2$ by a simple induction. No proof of that case is given. And the following generalization of it is stated, also without proof: if C, D_1, D_2 are in general position, then the appropriate envelope X has degree 24. Finally, the limit of X, as D_2 approaches a conic through $C \cap D_1$, is described, but again no proof is given.

S. L. Kleiman

From MathSciNet, March 2014

MR1662198 (2000d:14056) 14N05; 14H40

Previato, Emma

Poncelet's theorem in space.

Proceedings of the American Mathematical Society **127** (1999), *no.* 9, 2547–2556.

In this paper a generalization of Poncelet's theorem from plane configurations to configurations in the complex projective space $\mathbf{P}^{g+1}$ is presented.

The two equivalent versions of this theorem in $\mathbf{P}^{g+1}$ are written as follows: take a confocal family of quadrics $\mathcal{E}_\lambda$ $(\lambda \in \mathbf{P}^1)$ and a precise projective definition of reflection; choose $\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_{g+1}$ quadrics of the family.

Version I. If there is a polygon inscribed in $\mathcal{E}_0$ and circumscribed to each $\mathcal{E}_k$ so that its successive sides are linked by a reflection on $\mathcal{E}_0$, then there is a g-dimensional family of such polygons obtained by moving any vertex on $\mathcal{E}_0$.

Version II. By fixing $\mathcal{E}_k$ $(k = 1, \cdots, g)$ and choosing any number of confocal quadrics $\mathcal{E}_{\lambda_0}, \cdots, \mathcal{E}_{\lambda_n}$, one can assume to have a Poncelet polygon whose successive sides are obtained by bouncing on $\mathcal{E}_{\lambda_j}$, in which case there is a g-dimensional family of such n-gons obtained by moving any of the vertices p_j over $\mathcal{E}_{\lambda_j}$.

For the case of plane configurations, a very interesting interpretation of Poncelet's theorem is revisited [P. Griffiths and J. Harris, Comment. Math. Helv. **52** (1977), no. 2, 145–160; MR0498606 (58 #16695)]. An elliptic curve C and an order n point τ of C are the main ingredients of this nice interpretation. The idea to generalize the theorem for configurations of quadrics in $\mathbf{P}^{g+1}$ is to find a hyperelliptic curve X and a point p in its Jacobian, playing the role of C and τ in the classical Poncelet theorem. Explicit constructions for X and p are given.

With this description it is possible to bring together some different known versions of Poncelet-type theorems. Also, some directions for further investigations (arithmetic questions, vector bundles and compactifications) are suggested.

Roberto Munoz

From MathSciNet, March 2014

MR2964027 (Review) 14-02; 14-01

Dolgachev, Igor V.

Classical algebraic geometry.

A modern view.

Cambridge University Press, *Cambridge*, 2012, xii+639 pp pp., $150.00,
ISBN 978-1-107-01765-8

This is a long-expected book—preliminary versions have been circulating on the internet for years. As the title indicates, the aim of the book is to express in modern language some of the results in classical algebraic geometry obtained in the 19th (sometimes 20th) century. Some of these results can occasionally be found scattered in the modern literature, but this is certainly the first book devoted entirely to classical topics. Of course, in writing such a book one has to make choices. Still, the amount of material covered is absolutely impressive—it reflects the amazing culture of the author.

Here is a glimpse of the main topics of the book. Chapter 1 deals with polarity and apolarity. Polarity associates to a homogeneous polynomial F of degree d on a vector space E a symmetric d-linear form on E. Setting the first k variables equal to $a \in E$ gives the k-th polar hypersurface of F at a. The quadric polars ($k = d - 2$) lead to the definition of the *Hessian* and the *Steinerian* of F, two important hypersurfaces associated to F.

Apolarity is the classical terminology for the duality between $\mathsf{S}^d E^\vee$ and $\mathsf{S}^d E$. The book discusses a famous application to the problem of writing a general form F of degree d as a sum of d-th powers of linear forms ("Waring problem for forms").

Chapter 2 recalls the geometry of conics and quadrics. Many classical results (Chasles, Desargues, Pascal and Brianchon, Darboux, Steiner, Salmon ...) are interpreted in modern language. Some more sophisticated geometry appears here and there, for instance with the variety of self-polar triangles of a conic or the celebrated Poncelet porism.

Chapter 3 deals with plane cubics: the Hesse pencil and its automorphism group, the dual curve, the Hessian, the projective generation of plane cubics. The chapter concludes with a more advanced section (mostly without proof) on the invariant theory of plane cubics.

Chapter 4 discusses the representation of a form F as a determinant of linear forms. The case of *symmetric* determinants is particularly interesting since it is related to *theta characteristics* on the hypersurface $V(F)$. The case where $V(F)$ is a curve or a surface, possibly singular, is studied in detail.

Chapter 5 is devoted to theta characteristics on a smooth curve C, and to the corresponding theta functions. The study of theta characteristics gives rise to a rich geometry over $\mathbb{F}_2$ which has been much studied classically, with a flowery terminology: syzygetic (or azygetic) triad (or tetrad), Steiner complexes, fundamental sets.

All these notions are explained here in modern language. The last section explains the *Scorza correspondence* associated to a non-effective theta characteristic on a curve of genus g, and the rather mysterious Scorza quartic (a quartic hypersurface in $\mathbb{P}^{g-1}$) constructed from that correspondence.

In Chapter 6 the previous results are applied to plane quartic curves. The special feature is that odd theta characteristics correspond to the famous 28 bitangent lines to the quartic, so that general results on theta characteristics translate nicely into geometric statements on the bitangents. Choosing a fundamental set, or a line bundle of order 2, or an even theta characteristic, gives different types of equations for the curve, from which various geometric properties follow.

Next, some special quartics are studied: the Clebsch quartics, which can be written as a sum of 5 fourth powers, and the Lüroth quartics, which pass through the vertices of a complete pentagon. There is a beautiful relation between these two families, related to the Scorza correspondence introduced in Chapter 5.

The chapter concludes with a complete description of the possible automorphism groups of plane quartics.

Chapter 7 deals with Cremona transformations. After giving some general results and some interesting examples, the book focuses on planar transformations. This culminates with the Noether factorization theorem: the Cremona group is generated by the standard quadratic transformation and projective automorphisms.

Chapter 8 is devoted to del Pezzo surfaces. This will be more familiar to the modern reader, since there are a number of recent texts on the subject. The treatment here is fairly complete, with a detailed study of the geometry of those surfaces in each degree (except 3) and a complete description of the possible automorphism groups.

The special case of cubic surfaces is treated in Chapter 9. It starts with the classical results about lines on the surface, with emphasis on a particular configuration, the "double-sixes" and their remarkable properties (Schur's theorem). Next, a complete classification of singular cubics is given. Various ways of writing the equation of the surface, with some interesting geometric consequences, are explained: Cayley-Salmon equation as a determinant, Sylvester representation as a sum of 5 cubes, Cremona hexahedral equation. Again the chapter concludes with a complete description of the possible automorphism groups.

Chapter 10 studies the geometry of lines in projective space. After discussing the Grassmann variety of lines, the author considers divisors of degree 1 and 2 in this variety, called linear and quadratic complexes. The so-called "singular surface" of a quadratic complex of lines in $\mathbb{P}^3$ is a Kummer surface; it is shown that many properties of the Kummer surface can be explained from that point of view. Then the author discusses some particular quadratic complexes (harmonic, tangential, tetrahedral, ...).

The last section studies ruled surfaces, their numerical characters, and the relations between them ("Cayley-Zeuthen formulas"). As an application, a complete classification of quartic ruled surfaces in $\mathbb{P}^3$ is given.

Each chapter includes a large number of exercises, and a very interesting historical note with references (more than 600 in total). Most of the book is accessible to those with a standard background in algebraic geometry (as found for instance in R. Hartshorne's book [*Algebraic geometry*, Springer, New York, 1977; MR0463157 (57 #3116)]), though some parts may require some more specific knowledge, e.g., invariant theory or some advanced curve theory.

The author has rendered a great service to the algebraic geometry community: most of the material treated was available previously only in classical texts, which are quite difficult to read for modern mathematicians. This is a wonderful book. Anyone interested in classical algebraic geometry should have a copy.

Arnaud Beauville
From MathSciNet, March 2014