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Ricci flow for shape analysis and surface registration: theories, algorithms and applications, by Wei Zeng and Xianfeng David Gu, Springer Briefs in Mathematics, Springer, New York, 2013, xii+139 pp., ISBN 978-1-4614-8781-4

1. Introduction

One of the goals in the field of modern geometric analysis is to study canonical structures on manifolds and vector bundles—existence, uniqueness, and moduli. One of the pioneers of analytical approaches in this field is Yau, who, with Schoen and other coauthors solved a number of outstanding geometric problems, many of which are related to elliptic PDEs. A personal account of these developments is in the two-volume set [65]. Around the same time, there were remarkable developments in other areas of geometry. A more geometric theory, including the fundamental theories of compactness and collapse, was developed by Cheeger, Gromov, and others. Thurston revolutionized low-dimensional topology by developing a geometric theory aimed at understanding 3-manifolds via the geometrization conjecture and related ideas. Soon after, inspired by the earlier work of Eells and Sampson on harmonic maps, Hamilton invented the Ricci flow, a parabolic PDE, and through a number of very original works nearly single-handedly developed it into a compelling program to approach the Poincaré and geometrization conjectures. Two decades later, Perelman solved the Poincaré and geometrization conjectures by introducing a plethora of deep and powerful ideas and methods into Ricci flow to complete Hamilton's program. At the present time, more than another decade later, geometric analysis is a thriving field with many active subfields, including Kähler geometry, influenced by the work of Tian, Donaldson, Chen, and others, the study of Ricci curvature by Cheeger, Colding, Minicozzi, and others, geometric flows such as the mean curvature flow by Huisken, White, and others, and the work of Brendle to solve long-standing conjectures, to just name a few of the many directions this field has taken.

The idea that a discrete set of values determines geometry is the main tenet of *discrete differential geometry*. These ideas come from a number of disparate directions. Whitney used numerical analysis to connect de Rham theory and simplicial homology [64], and these ideas form the central tenets of *discrete exterior calculus* [20]. In a separate intellectual stream, Regge introduced a notion of discrete geometry that he considered a way of developing computational and quantum gravity from first principles that were motivated by, but not dependent on, the theory of smooth manifolds (see [34, 51]). Thurston introduced the idea of circle packings for approximating Riemann mappings, and this developed into a whole field of circle packing and related topics (see Stephenson's book [58], reviewed in [6]). A discrete form of integrable systems was developed by Bobenko and Suris [5] into a geometric theory, while other groups [1, 23] have developed similar or competing theories arising from numerical analysis and mathematical physics. All of these directions may be characterized as structure-preserving discretizations.

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The background for each of these is a notion of geometry that parallels that of Riemannian geometry. Each piecewise constant curvature space is defined by local pieces whose geometry is determined by finitely many parameters. Usually, this is a simplex whose edge lengths are specified and they uniquely determine a constant curvature simplex. The choice of which constant curvature is used in the simplex is part of the parameters; while zero curvature Euclidean simplices are easy to work with, it is sometimes desirable to work with hyperbolic or spherical simplices. Using Euclidean, hyperbolic, and spherical trigonometry, the angles of the simplex are determined by the edge lengths, as is the volume of each of its faces. If the space is a triangulated manifold, the specification of edge lengths and curvatures of the simplices completely determines a geometry on the manifold in the same way a Riemannian metric determines a geometry, namely by taking the distance between two points to be the infimum of lengths of paths.

Among the important applications of discrete differential geometry, one can mention, for instance, image analysis, network theory, and computer graphics. There is also a large community of mathematicians working from the perspectives of hyperbolic geometry, topology, dynamical systems, and complex analysis. The interplay here is a great example of a place where mathematicians, applied mathematicians, and computer scientists can find common ground to work on interesting and complex problems. The book is devoted mainly to certain applications in image analysis, but we will first discuss some context from the perspective of geometric analysis.

2. Uniformization and the Riemann mapping theorem

A forerunner in the search for canonical metrics is the uniformization theorem of complex analysis, which implies that in the conformal class of any closed Riemannian surface there exists a constant curvature metric. This can be formulated as a Riemannian geometry theorem as follows.

Theorem 1 (Uniformization)**.** *Let (Σ, g) be a closed surface with Riemannian metric g. Then there exists a function $f \in C^{\infty}(\Sigma)$ such that $e^{f} g$ has constant curvature.*

Since the Gauss–Bonnet theorem states that $\int_{\Sigma} K dA = 2\pi\chi_{\Sigma}$, where K is the Gaussian curvature of (Σ, g), dA is the area measure, and χ_{Σ} is the Euler characteristic of Σ, the sign of the constant curvature is determined by the topology of the surface. By scaling the metric, it follows that any surface has a constant curvature metric with curvature -1, 0, or 1. Using this, one can turn the problem of finding a constant curvature metric into a partial differential equation for the function f, namely $\triangle f - K + \varepsilon e^{2f} = 0$, where $\varepsilon = \pm 1$ or 0, the target constant curvature, and $\triangle$ is the Laplacian operator associated with (Σ, g).

A related problem for domains in the plane is resolved by the Riemann mapping theorem.

Theorem 2 (Riemann Mapping Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a proper, open, simply connected subdomain of the plane. Then there exists a conformal diffeomorphism from the unit disk $\phi : \mathbb{D} \to \Omega$ that is unique up to precomposition with Möbius transformations.*

It is of note that both the Uniformization theorem and the Riemann mapping theorem can be proven using geometric flows ([8, 12, 35]).

3. Circle packing, hyperbolic geometry, and topology

Andreev and later Thurston related the construction of hyperbolic structures on 3-manifolds with patterns of circles in the plane. This was also previously described by Koebe for the sphere. A good summary of the study of such circle packings is in [6]. The main connection is that circles in the plane correspond with planes in the half-space model of hyperbolic 3-space, and so configurations of circles correspond to hyperbolic polyhedra. Similarly, there is a correspondence between triangulations in the plane and hyperbolic tetrahedra by considering each edge of a Euclidean triangle as the projection from infinity of an ideal hyperbolic tetrahedron with fourth vertex at infinity. The interplay between hyperbolic tetrahedra (classical, ideal, and hyperideal), hyperbolic volume, circle packings, and angles of triangles has been well studied [4, 40, 43, 46, 52, 55, 56]. These volume structures allow one to relate angles and lengths in important ways. For instance, given a topologically triangulated domain with angles assigned to the triangles so that the sum in a triangle is equal to π (so the triangle would be Euclidean), can you find edge lengths so that the corresponding Euclidean triangles fit together and have the assigned angles when calculated using the law of cosines? Yes—this was proven by Rivin in [52] using the volume of corresponding ideal tetrahedra.

The connection between angle structures and the topology of 3-manifolds has important ramifications. In particular, a proposed alternative proof of the Poincaré conjecture is described in [44] and [45]. The main idea is that given a minimal triangulation of a hyperbolic manifold with a single vertex (these always exist), then it is conjectured that either the triangulation satisfies an equation called the Thurston equation or it satisfies Haken's normal surface equation of a particular type. If it satisfies Haken's normal surface equation, then it can be classified, and the only simply connected possibility is the 3-sphere. If it satisfies Thurston's equation, then it cannot be simply connected. Hence, the Poincaré conjecture would be solved. However, the current best result is only that either the triangulation satisfies a generalized form of Thurston's equation or the Haken equation condition.

4. Curvature and Laplacian

Two central elements in geometric analysis are the curvature and Laplacian. The Gauss–Bonnet formula relates the curvature of a Riemannian surface, the geodesic curvature of the smooth part of its boundary, and the angle deficit of its corners in a beautifully succinct way:

$$\int_\Sigma K dA + \int_{\partial\Sigma} \kappa ds + \sum_i (\pi - \theta_i) = 2\pi,$$

where κ is the geodesic curvature, ds is the arclength measure, and θ_i is the interior angle at a point. If one restricts to surfaces with totally geodesic boundary, this formula relates angles deficit with the integral of the Gauss curvature. In discrete differential geometry, we usually consider only constant curvature geodesic triangles, and so we have a relationship between area, curvature, and topology similar to the usual Gauss–Bonnet theorem. Connections between angles and curvature can also be seen by comparing the volume of tubes formulas, by comparing Chern's kinematic formulas, and by looking at asymptotics of the heat kernel (see [9–11, 49]).

On a surface, it is quite natural to take the curvature K_v at a vertex to be 2π minus the sum of the interior angles at that point (or some normalized version of this). At first inspection, one sees that positive, zero, and negative curvature have similar interpretations to those in the theory of smooth surfaces. The question is, do these discrete curvatures concentrated at vertices of a piecewise constant curvature manifold converge to the Gauss curvature? A special case of a theorem of Cheeger, Müller, and Schrader [10] shows that if we take an appropriate sequence of geodesic triangulations of a Riemannian manifold (in any dimension), convert the simplices into Euclidean simplices, and study the convergence of the certain discrete curvatures (coming from angles), they will converge to certain curvature measures (Lipschitz–Killing curvatures) on the Riemannian manifold.

The Laplacian as an operator on functions defined on vertices has broad appeal in graph theory, for instance the famous matrix-tree theorem, which related the product of nonzero eigenvalues of the Laplacian to counting spanning trees. Various notions from geometric analysis have been applied to graphs and graph Laplacians, leading to graph invariants such as isoperimetric inequalities, Cheeger constant/inequality, and the Colin de Verdiere number (see [16, 18, 59] and subsequent work for a nice summary). In the context of discrete differential geometry, the Laplacian can refer to any operator of the form

$$\triangle f_v = \sum_u c_{uv} \left(f_u - f_v \right), \tag{1}$$

where the sum is over all vertices u and c_{uv} is a (usually nonnegative) coefficient of the edge uv that is zero if uv is not an edge. If $c_{uv} \geq 0$, then we can say that the Laplacian satisfies a weak maximum principle, which says the following. If $v_m = \arg\min \{ f(v) : v \in V \}$ and $v_M = \arg\max \{ f(v) : v \in V \}$, then we have

$$\triangle f_{v_m} \geq 0 \quad \text{and} \quad \triangle f_{v_M} \leq 0$$

at a minimum and at a maximum of f, respectively. If $c_{uv} = c_{vu}$, then we can also use summation by parts to see that

$$\sum_v g_v \triangle f_v = \frac{1}{2} \sum_{uv} c_{uv} \left(g_u - g_v \right) \left(f_u - f_v \right). \tag{2}$$

It turns out that in many applications of discrete differential geometry, there is a natural choice for the coefficients c_{uv}, often arising as a ratio of an area on the Poincaré dual of the edge uv to the length of that edge. In discrete exterior calculus (see [20]), an entire discrete Hodge theory is defined, which, in particular, allows the definition of a gradient and divergence that results in a Laplacian of the type in (1). Such a Laplacian also arises in certain variations of curvature formulas, such as in [14, 47]. The work of He [36] and subsequent work [21, 27, 31] describe the connection between the two forms of the Laplacian. The main point is that a geometric structure is necessary on the Poincaré dual of the triangulation. This dual structure is usually defined by giving centers to each simplex, and then defining the successive simplices by connecting the centers. One notes that this is in line with the usual interpretation of the coefficients as conductivity, as it is inversely proportional to distance and directly proportional to cross sectional area. When the centers of a triangle are prescribed to be the center of the circumcircle, one arrives at the well-known cotan Laplacian ([22, 50]), which is also the finite element Laplacian.

5. Application of curvature flow

Given a triangulation, we can assign edge lengths and choose a constant curvature geometry (Euclidean, hyperbolic, or spherical), called the background geometry, to produce a metric structure. If it has zero curvature at the vertices (i.e., the angles sum to 2π at each of the vertices), then the surface has been given a geometric structure that can be locally embedded into the background geometry. This allows one to convert embedding problems into prescribed curvature problems, and many of the applications in the field, including most of the applications in the book, come from this idea. Often the zero curvature geometry has some sort of uniqueness property as well, due to connections to discrete Laplacians.

One basic embedding problem is the following: given a simply connected domain in the plane, find another domain that is embedded in the disk with some appropriate boundary condition that makes it fill the disk in some way (for instance, all boundary vertices are on the unit circle or all boundary circles are tangent to the unit circle). By the Riemann mapping theorem, we know that every domain can be essentially uniquely mapped to the disk by a conformal transformation. In discrete differential geometry, we can ask the same question for a discrete conformal structure. This is precisely what is proven by Rodin and Sullivan in [54] (also described in [58]; see also He and Schramm [37] for a proof that does not presuppose the Riemann mapping theorem).

In the context of Riemannian geometry, a conformal deformation of a metric is a rescaling of the metric piecewise without regard to direction; one can describe a conformal deformation of a metric g as a one parameter family of positive functions $f(t)$ giving the deformation $g(t) = f(t)\,g$. In the context of discrete differential geometry, the geometry is specified not by a Riemannian metric but by a collection of lengths together with a constant curvature background. If the lengths were determined by parameters at vertices, then by changing these parameters, one could create a similar notion of conformal deformation by changing those vertex parameters. One important example is the circle packing formulation, where each vertex has a radius r_v and each edge uv has length $\ell_{uv} = r_u + r_v$. We can imagine for every vertex in a triangle uvw that we have a circle centered at each vertex with the circles mutually tangent. In this way, the function r on the vertices determines the length and hence the whole geometry. By changing the values of r at different vertices, we are changing the geometry, but only in a way that is the same in all directions. In this way, the deformation is similar to that of the conformal deformation of a metric.

There are many other ways to assign parameters to the vertices that determine the length. A large class of these was studied in [27], and it was shown in [32] (see also [66]) that all of these conformal deformations can be described by certain functions. Previously, different discrete conformal deformations were considered in [3, 41, 47, 57] and many other places. In general, we consider a variation that makes ℓ_{uv} dependent on weights f_u and f_v that are assigned to the vertices of the edge uv. A conformal variation then changes lengths by $\frac{d}{dt}\ell_{uv} = \frac{\partial \ell_{uv}}{\partial f_u}\frac{df_u}{dt} + \frac{\partial \ell_{uv}}{\partial f_v}\frac{df_v}{dt}$.

The main geometric property that makes conformal variations especially useful is that the variation of the curvature gives a Laplacian of the variations

$$\frac{dK_v}{dt} = -\triangle \frac{df_v}{dt} \tag{3}$$

in the Euclidean background, and there is an extra linear reaction term in the other constant curvature backgrounds (see [31] for Euclidean background and [32] and [66] for other constant curvature backgrounds). Thus the stability of the zero curvature problem is related to the definiteness of the Laplacian. This is true for many, though not all, situations since the coefficients c_{uv} may have to be negative for (3) to be valid.

The variation formula (3) indicates that evolution of f_v by minus the curvature induces a (time dependent) heat equation on the curvature. Such a formula is similar to evolutions by curvature in Riemannian geometry such as the Ricci flow and Yamabe flow. This and similar curvature evolutions lead to an averaging of the curvatures and are a way of finding canonical metrics. In the smooth Ricci flow in two dimensions, the flow induces a reaction-diffusion equation on the curvature, namely $\frac{\partial R}{\partial t} = \triangle R + R^2$. One can rewrite this in terms of the curvature measure as

$$\frac{\partial}{\partial t}\left(RdA\right) = \triangle RdA.$$

Similarly, it follows from (3) that the discrete Ricci flow (combinatorial/discrete surface Ricci flow) with Euclidean background gives the evolution

$$\frac{dK_v}{dt} = \triangle K_v.$$

There are similar formulas of induced reaction-diffusion equations with spherical and hyperbolic backgrounds (see, e.g., [14]).

An important fact in two dimensions is that the Ricci flow evolution is, in fact, a gradient flow, with a functional first described by Colin de Verdiere in [17]. This is also true in two dimensions for the smooth Ricci flow [13]. The connection between the smooth and discrete functionals in not well understood, though there is some suggestion of a connection in [40].

In the discrete setting, however, this has an additional important consequence: Newton's method is a good alternative to the Ricci flow. Newton's method has much faster convergence than the gradient flow, and since the Hessian of the curvature functional is well understood, the use of Newton's method on closed manifolds is quite easy. This creates a great improvement on the run-time of uniformizing manifolds with many triangles as compared to previous methods such as Thurston's algorithm as implemented by Stephenson (see [58]).

6. Three dimensions

An extensive understanding of prescribed geometry in three dimensions could lead to interesting applications for volumetric modeling or to the study of 3-manifold geometry. Discrete differential geometry was used to find an embedding of a piecewise Euclidean 2-sphere with positive vertex curvature into $\mathbb{R}^3$ by Bobenko and Izmestiev [2], giving an iterative proof of Alexandrov's theorem. Hyperbolic embedding theorems from dihedral angle data had been treated by Hodgson and Rivin [39] and more recently [24, 25]. Sphere packing metrics have been studied by a number of authors [15, 26–28], and there is some general theory on angle variations in three-dimensional piecewise Euclidean manifolds in [31] and further work on hyperbolic manifolds in [60]. Curvature flow on hyperbolic 3-manifolds with totally geodesic boundary was also studied in [42]. An alternative approach of simplicial Ricci flow was proposed in [48] and successive papers.

7. The book

The book is aimed at providing mathematical foundations and algorithms for practitioners, especially engineers, who may find themselves in need of certain techniques derived from discrete differential geometry. For this reason, it is somewhat less technical than the related works [33] and [19].

The book covers diverse topics, including basic Riemannian geometry, algebraic topology, Riemann surfaces, quasiconformal maps and harmonic maps, Teichmüller spaces, and algorithmic details. Chapter 4 is the main theme of the book, which covers the discrete Ricci flow on surfaces. It introduces Thurston's circle packing theory and its variational proofs and is built on the work of the first and the third reviewers. Complete proofs for most of the theorems and lemmas are given in detail in this chapter.

One of the most fascinating parts of the book consists of the applications of discrete Ricci flow pioneered by the authors. It covers many topics, including surface registration and tracking, isometric and conformal mapping, shape analysis, and surface conformal registrations. The authors have developed very effective methods of computing the solutions of discrete Ricci flow and have applied them successfully to solve many real world imaging problems. The basic idea behind the thinking is the following. A curved surface in Euclidean 3-space is difficult for computation. For instance, it is not easy to compare two simply connected surfaces in a 3-space up to isometry. By using the Ricci flow, one sends each surface to the unit disk conformally and records the curvature information as a function on the unit disk. Then it is easier to check if there is a disk preserving Möbius transformation sending one curvature function to the other. Furthermore, conformal mappings distort two-dimensional shapes at small scales up to scaling and rotation; thus some of the main features of the shape can still be preserved. This is the basis of the application of discrete Ricci flow for colon flattening (see Figure 5.7 on page 120 of the book). The goal is to find precancerous polyps using virtual colonoscopy. Using the Ricci flow, one sends the CT images of the colon wall conformally to planar rectangles. Then the polyps and other abnormalities can be found efficiently on the planar image by Computer Aided Detection techniques. This is a part of automated diagnosis in medical science, where computer-aided diagnosis is used by physicians for the interpretation of medical images. There are many other applications of discrete surface Ricci flow technology discussed in the book. It is highly recommended that the reader browse through Chapter 5 to see these applications. We believe that many more applications of the discrete Ricci flow are yet to be discovered.

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