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SELECTED MATHEMATICAL REVIEWS

related to the first article in the previous section by
W. T. GOWERS

MR1631259 (2000d:11019) 11B25; 11N13

Gowers, W. T.

A new proof of Szemerédi's theorem for arithmetic progressions of length four.

Geometric and Functional Analysis **8** (1998), *no.* 3, 529–551.

This remarkable paper gives a new proof that every subset of the integers with positive density must contain arithmetic progressions of length four. This was conjectured by P. Erdős and P. Turán [J. London Math. Soc. **11** (1936), 261–264; Zbl 015.15203], and eventually proved by E. Szemerédi [Acta Math. Acad. Sci. Hungar. **20** (1969), 89–104; MR0245555]. Szemerédi's proof depended on the theorem of van der Waerden. It gave no explicit function $\eta(N)$, tending to zero with N, such that an integer subset of $[1, N]$ of size at least $\eta(N)N$ must contain a progression of length 4. Since then it has been an important open problem, which this paper solves, to prove Szemerédi's result with an explicit function $\eta(N)$. The result given here is that there is a positive constant c such that any integer subset of $[1, N]$ with at least $N(\log\log\log N)^{-c}$ elements contains an arithmetic progression of length 4. In a future paper it is promised that the number of logarithms will be reduced to 2. Moreover, the author plans to extend the technique to arithmetic progressions of arbitrary length $k \geq 4$, obtaining bounds of the same form, but with the constant c depending on k.

The initial stages of the proof are motivated by K. F. Roth's treatment [J. London Math. Soc. **28** (1953), 104–109; MR0051853] of progressions of length 3, in which the circle method was used. However, it is clear that the usual notion of uniform distribution is not sufficient to handle progressions of length 4. Instead Gowers uses "quadratic uniformity". If $A \subseteq [1, N]$ is an integer set with ηN elements, and characteristic function χ_A, define $a_n = \chi_A(n) - \eta$. Roughly speaking, one says that A is quadratically uniform if

$$\sum_{n \leq N} a_n a_{n+k} \exp(2\pi i n\theta) = o(\eta N),$$

uniformly in θ, for "almost all" relevant k. It is shown that a quadratically uniform set contains arithmetic progressions of length 4.

For a set that fails to be quadratically uniform, one can find many values of k for which the above sum is large at some value $\theta = \gamma(k)/N$, with $\gamma(k)$ integral. The proof proceeds to find a large set $\mathcal{K}$ of values of k for which the graph $\Gamma = \{(k, \gamma(k)) \colon k \in \mathcal{K}\}$ has a difference set whose cardinality is little more than that of Γ itself. This is the situation described by the theorem of G. Freiman [*Foundations of a structural theory of set addition*, Translated from the Russian, Amer. Math. Soc., Providence, R. I., 1973; MR0360496]. A quantitative version of this result, due to I. Z. Ruzsa [Acta Math. Hungar. **65** (1994), no. 4, 379–388; MR1281447], is applied to show that there is a long arithmetic progression, almost all of whose elements lie in Γ. Using this information it is shown that if A is not quadratically uniform, then there exist α and β, and a long arithmetic progression P, such that

$$\sum_{n\in P} a_n \exp\bigl(2\pi i(\alpha n^2+\beta n)\bigr)$$

is large. The proof is now completed by showing that there is a further large arithmetic progression Q, say, such that $\sum_{n\in Q} a_n$ is large and positive. It follows that the density η for the original set A must be appreciably smaller than for $A\cup Q$. As in Roth's theorem, one can now iterate this fact to get the bound $\eta \ll (\log\log\log N)^{-c}$.

It is natural to ask to what extent the proof can be adapted to attack the problem of 4 primes in arithmetic progression. In fact, large parts of the argument go through. However, it is clear that the use of anything like Freiman's theorem will provide bounds which are too weak for such an application.

D. R. Heath-Brown

From MathSciNet, October 2016

MR2150389 (2007b:37004) 37A05; 28D05

Host, Bernard; Kra, Bryna

Nonconventional ergodic averages and nilmanifolds.

Annals of Mathematics. Second Series **161** (2005), *no.* 1, 397–488.

The paper under review makes a landmark contribution to the literature surrounding the nonconventional ergodic averages introduced by H. Furstenberg in his ergodic-theoretic proof of Szemerédi's theorem [E. Szemerédi, Acta Arith. **27** (1975), 199–245; MR0369312].

Let $(X,\mathcal{A},\mu,T)$ be a finite-measure, invertible measure preserving system. Furstenberg showed in [J. Analyse Math. **31** (1977), 204–256; MR0498471] that for any natural number k and any non-negative, non-identically-zero $f\in L^\infty(X)$, one has

$$\liminf_{n-m\to\infty} \frac{1}{n-m}\sum_{i=m}^{n-1}\int f(x)f(T^i x)f(T^{2i}x)\cdots f(T^{ki}x)\,d\mu(x)>0.$$

Furstenberg also showed that the limit exists for $k=2$ by proving

$$\lim_{n-m\to\infty}\frac{1}{n-m}\sum_{i=m}^{n-1} f_1(T^i x)f_2(T^{2i}x)$$

to exist in $L^2(X)$ for bounded f_1, f_2. A proof of the existence of the corresponding limit for the ($k=3$, T totally ergodic) case was given in a series of papers by J.-P. Conze and E. Lesigne [cf. Bull. Soc. Math. France **112** (1984), no. 2, 143–175; MR0788966]; the means for proving the general $k=3$ case is perhaps implicit in [H. Furstenberg and B. Weiss, in *Convergence in ergodic theory and probability (Columbus, OH, 1993)*, 193–227, de Gruyter, Berlin, 1996; MR1412607], though the authors' [Ergodic Theory Dynam. Systems **21** (2001), no. 2, 493–509; MR1827115] contains the first fully general, published proof for $k=3$. The present paper establishes the existence of an L^2-limit for general k.

Significantly, the authors consider another type of ergodic averaging which had only recently gained attention: the so-called averaging along cubes. The cube averaging scheme evolved out of methods devised by W. T. Gowers in his proof of Szemerédi's theorem via harmonic analysis [Geom. Funct. Anal. **11** (2001), no. 3, 465–588; MR1844079]. The first to import these structures into the ergodic-theoretic context under consideration was V. Bergelson, who in [*Descriptive set theory and dynamical systems (Marseille-Luminy, 1996)*, 31–57, Cambridge Univ. Press, Cambridge, 2000; MR1774423] showed that

$$\lim_{\substack{n_i - m_i \to \infty \\ i=1,2}} \frac{1}{(n_1 - m_1)(n_2 - m_2)} \sum_{i_1=m_1}^{n_1-1} \sum_{i_2=m_2}^{n_2-1} f(T^{i_1}x)g(T^{i_2}x)h(T^{i_1+i_2}x)$$

exists in $L^2(X)$ for bounded functions f, g, h. It follows from this proof that for a set A of positive measure, $\mu(A \cap T^n A \cap T^n A \cap T^{n+m} A)$ has average value approaching at least $\mu(A)^4$ over sufficiently large rectangles in $\mathbf{Z}^2$. One can see this as case $k = 2$ in a more general scheme, case $k = 1$ consisting in the ergodic theorem and Khinchin's recurrence theorem. The authors [in *Modern dynamical systems and applications*, 123–144, Cambridge Univ. Press, Cambridge, 2004; MR2090768] proved case $k = 3$ of this scheme, in which one considers a triple Cesàro limit of terms which are products of seven functions evaluated along vertices of 3-dimensional parallelepipeds or "cubes"; here they establish the general case.

The paper follows the usual global strategy for establishing limits for non-conventional ergodic averages; namely identification of a characteristic factor, followed by a refined analysis on this factor. For example, a factor $\mathcal{Z} \subset \mathcal{A}$ (here $\mathcal{Z}$ is a T-invariant σ-algebra) is characteristic for the averaging scheme $\lim_{n-m\to\infty} \frac{1}{n-m} \sum_{i=m}^{n-1} f_1(T^i x) f_2(T^{2i} x) \cdots f_k(T^i x)$ if the limit reduces to zero 1 whenever $E(f_i|\mathcal{Z}) = 0$ for some i. It is easy to see that if $\mathcal{Z}$ is characteristic, then in establishing existence of the limit, one may assume without loss of generality that the f_i are $\mathcal{Z}$-measurable. In his proof of Szemerédi's theorem, Furstenberg showed that the maximal $(k-1)$-step distal factor is characteristic for the averaging scheme in question; when $k = 2$ this is the Kronecker factor, on which one can easily compute the limit, but already for $k = 3$ it is too fine to readily establish existence of the limit, which in fact lies on the coarser Conze-Lesigne algebra. The characteristic factors $\mathcal{Z}_k$ found by the authors for the aforementioned averaging schemes are indeed the Kronecker and CL-factors for the appropriate small values of k. Having constructed them, the bulk of the work here is in establishing that for general k, the restriction of the system to $\mathcal{Z}_k$ is an inverse limit of translations on nilmanifolds. Since existence of the corresponding limit was previously known in this setting (cf. [A. Leibman, Ergodic Theory Dynam. Systems **25** (2005), no. 1, 201–213; MR2122919] or [T. Ziegler, Ergodic Theory Dynam. Systems **25** (2005), no. 4, 1357–1370; MR2158410]), this suffices for the proof.

In constructing the factors $\mathcal{Z}_k$, the authors follow Furstenberg's original proof of the Szemerédi theorem in considering conditional product measures on product spaces. The novelty here is that instead of considering measures on X^k, they construct measures on the "cube product" X^{2^k}; elements explored exhaustively in [W. T. Gowers, op. cit.] force relevance to averages taken on arithmetic progressions. Unfortunately the Gowers connection, though acknowledged, is not explained (its sole mention serves to de-emphasize the connection, if anything), rendering many of the early moves in the paper somewhat opaque and unmotivated. Yet it is easy to give a brief motivating synopsis, done here for $k = 3$.

The Gowers uniformity seminorm $\|\!| \cdot |\!\|_3$ is defined for real-valued functions on Z_N by

$$\|\!| f |\!\|_3 = \Big(\sum_{a,b,c,d \in Z_N} f(a)f(a+b)f(a+c)f(a+b+c) \times f(a+d)f(a+b+d)f(a+c+d)f(a+b+c+d) \Big)^{1/2^3}.$$

Taking $X = \mathbf{Z}^N$ with counting measure and letting T be the shift, this is just

$$\left(\sum_{b,c,d=0}^{N-1} \int_X fT^b fT^c fT^{b+c} fT^d fT^{b+d} fT^{c+d} fT^{b+c+d} f \, d\mu \right)^{1/2^3}.$$

Letting now (X, μ, T) be any ergodic system and normalizing so that $\mu(X) = 1$, one can start with this expression and pass to Cesàro limits in the variables d, c and b in turn. Passing to a Cesàro limit in d, one obtains

$$\left(\frac{1}{N^2} \sum_{b,c=0}^{N-1} \int_{X^2} (f \otimes f)(T \times T)^b (f \otimes f)(T \times T)^c \times (f \otimes f)(T \times T)^{b+c} (f \otimes f) \, d\mu^{[1]} \right)^{1/2^3},$$

where $\mu^{[1]} = \mu \times \mu$. Now passing to a Cesàro limit in c, one gets

$$\Big(\frac{1}{N} \sum_{b=0}^{N-1} \int_{X^2} E\big((f \otimes f)(T \times T)^b (f \otimes f) | \mathcal{I}^{[1]}\big)^2 \, d\mu^{[1]} \Big)^{1/2^3} = \Big(\frac{1}{N} \sum_{b=0}^{N-1} \int_{X^4} (f \otimes f \otimes f \otimes f)(T \times T \times T \times T)^b \times (f \otimes f \otimes f \otimes f) \, d\mu^{[2]} \Big)^{1/2^3},$$

where $\mathcal{I}^{[1]}$ is the $\mu^{[1]}$-invariant algebra on $X \times X$ and $\mu^{[2]}$ is the conditional product of $\mu^{[1]}$ with itself over $\mathcal{I}^{[1]}$. Finally, upon passing to a Cesàro limit in b, one obtains

$$\Big(\int_{X^4} E(f \otimes f \otimes f \otimes f | \mathcal{I}^{[2]})^2 \, d\mu^{[2]} \Big)^{1/2^3} = \Big(\int_{X^8} (f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f) \, d\mu^{[3]} \Big)^{1/2^3},$$

where $\mathcal{I}^{[2]}$ is the $\mu^{[2]}$-invariant algebra on X^4 and $\mu^{[3]}$ is the conditional product of $\mu^{[2]}$ with itself over $\mathcal{I}^{[2]}$. The authors take this last expression as the definition of $\|\!| f |\!\|_3$. Higher values of k get a similar treatment by induction. The characteristic factors $\mathcal{Z}_k$ are in turn constructed out of the measurable structures $\mu^{[k]}$. The details are too involved for a review; we mention only that the uniformity seminorms are used to establish the property of being characteristic, the point being that when $\|\!| f_i |\!\|_k$ is small for some i this forces $\limsup_N \| \frac{1}{N} \sum_{n=0}^{N-1} (\prod_{i=1}^k T^{in} f_i) \|$ to be small.

More recently, a second proof of this paper's main result has been obtained in [T. Ziegler, J. Amer. Math. Soc. **20** (2007), no. 1, 53–97 (electronic); MR2257397 (2007j:37004)].

REVISED (December, 2006; January, 2007)

Randall McCutcheon
From MathSciNet, October 2016

MR2289012 (2008a:11002) 11-02; 05-02, 05D10, 11B13, 11P70, 11P82, 28D05, 37A45

Tao, Terence; Vu, Van

Additive combinatorics. (English)

Cambridge Studies in Advanced Mathematics, 105.
Cambridge University Press, *Cambridge*, 2006, xviii+512 pp., $85.00,
ISBN 978-0-521-85386-6; 0-521-85386-9

The subject of the book under review is additive combinatorics—a young and extensively developing area in mathematics with many applications, especially to number theory. Roughly speaking, one can define this area as combinatorics related to an additive group structure. Modern additive combinatorics studies various groups, from the classical group of integers to abstract groups of arbitrary nature.

It is difficult to determine a starting point for additive combinatorics. Among the origins of the theory one should mention the Cauchy theorem on set addition on the group of residues modulo a prime [A. L. Cauchy, J. École Polytech. **9** (1813), 99–116; per bibl.], I. Schur's theorem on monochromatic solutions to the equation $x+y=z$ [Jahresber. Deutsch. Math.-Verein. **25** (1916), 114–117; JFM 46.0193.02], and, certainly, the famous van der Waerden theorem on monochromatic arithmetic progressions [B. L. van der Waerden, Nieuw Arch. Wisk. **15** (1927), 212–216; JFM 53.0073.12]. Probably the first serious application of combinatorial methods to classical number theory was made by Shnirel′man. Using Brun's lower estimates for the density of $\mathbb{P}+\mathbb{P}$ where $\mathbb{P}$ is the set of primes he deduced that every integer > 1 is a sum of a bounded number of primes [L. Shnirel′man, Izv. Donsk. Politeh. Inst. **14** (1930), 3–28; JFM 56.0892.02; Math. Ann. **107** (1933), 649–690; Zbl 0006.10402].

Van der Waerden's theorem had a great influence on the development of additive combinatorics. In this connection, it is worthy of mention that the most spectacular results of additive combinatorics, namely, Szemerédi's theorem on arithmetic progressions in subsets of the set of integers of positive density, Gowers' estimates for the density of sets without arithmetic progressions, and, of course, the theorem of Green and Tao on the existence of arbitrarily long progressions in the set of primes, are directly related to van der Waerden's theorem. The last two results—and also such outstanding achievements as the theorem of Bourgain, Katz, and Tao on sums and products of sets in finite fields and Ruzsa-Chang's refinement of Freiman's theorem—have led to the extremely active development of additive combinatorics in the last decade. During this period it has become a very rich and fruitful theory that is interacting and interlacing different areas of mathematics, such as harmonic analysis, graph theory, probability theory, ergodic theory, geometry of numbers, and algebraic geometry. This theory is beautiful and contains a lot of challenging problems. It is not a surprise that it has combined the efforts of many leading mathematicians, including the authors of the book under review. However, there has been an absence of systematic exposition of contemporary additive combinatorics (earlier results are presented in the monograph by M. B. Nathanson [*Additive number theory*, Springer, New York, 1996; MR1477155]). The purpose of the book under review is to fill this gap.

The monograph is designed for a wide mathematical audience and does not require any specific background from a reader. However, everybody who intends to read this book should be ready to study tools and ideas from different areas of mathematics, which are concentrated in the book and presented in an accessible, coherent, and intuitively clear manner and provided with immediate applications to problems in additive combinatorics. The text is supplemented by a large number of exercises.

In Chapter 1 Tao and Vu discuss the well-known probabilistic method [see N. Alon and J. H. Spencer, *The probabilistic method*, Second edition, Wiley-Intersci., New York, 2000; MR1885388] and its applications to problems in additive number theory such as the construction of sum-free sets and thin bases.

In the next chapter general inequalities on doubling constants, Ruzsa's and Green's covering lemmas, the theorem of Balog-Szemerédi-Gowers, elementary sum-product estimates and Ruzsa's triangle inequality are considered. Also, non-commutative analogues of the results obtained are discussed.

In Chapter 3 some tools of geometry of numbers are described (for example, the Brunn-Minkowski inequality on addition of sets in $\mathbb{R}^d$). Applications of harmonic analysis are considered in the next chapter. In particular, the spectrum of additive sets, Bohr sets, and dissociated subsets of the spectrum are considered. Also, some applications to problems concerning $B_k[g]$ sets and progressions in sumsets are obtained.

The general theory of set addition in arbitrary groups is considered in Chapter 5. Tao and Vu give the definitions and obtain the properties of Freiman homomorphisms and the so-called e-transform. Theorems of Cauchy-Davenport, Kneser, Mann, Vosper and the very important Freiman theorem on the structure of sets with small doubling are proved.

In the next chapter the authors discuss some aspects of Ramsey theory and applications of graph theory to additive combinatorics. Also, the beautiful Plünnecke theorem on the connection between the cardinalities of $|A + B|$ and $|A + kB|$ is proved.

In Chapter 7 the direct and inverse Littlewood-Offord problems are studied by the Fourier approach and the probability method. The quadratic generalizations of the problem are considered.

Further, Tao and Vu discuss some problems of incidence, particularly, crossing numbers and the theorem of Szemerédi-Trotter. Also, they describe some applications of the Szemerédi-Trotter theorem to the sum-product problem in $\mathbb{R}$ and other fields.

In the next chapter the authors deal with methods of algebraic geometry—for example, the polynomial method—and applications of these methods to problems concerning addition with restrictions. Davenport's problem and the one- and multidimensional Erdős-Ginzburg-Ziv problems are considered.

In Chapter 10 Tao and Vu give several proofs of Roth's theorem on sets without arithmetic progressions using methods of harmonic analysis, the dynamical approach, and the original combinatorial method of Szemerédi. In the next chapter the case of arithmetic progressions of length greater than three is considered.

Gowers' method, the ergodic approach, and the hypergraph methods of proving the result are explained. Moreover, a sketch of the proof of Green-Tao's theorem on arithmetic progressions in the primes is given.

The last part of the book contains some results of Lev, Sarkozy, Szemerédi, and Vu on arithmetic progressions in sumsets and also theorems on complete and subcomplete sequences.

Sergeĭ V. Konyagin and Ilya D. Shkredov
From MathSciNet, October 2016

MR2680398 (2011j:11177) 11N13; 11B30, 11P32

Green, Benjamin; Tao, Terence

Linear equations in primes.

Annals of Mathematics. Second Series **171** (2010), *no.* 3, 1753–1850.

The paper under review is a landmark contribution to analytic number theory. The authors establish, conditioned on two conjectures labeled MN(s), GI(s), a vast multilinear generalization of the classical Dirichlet theorem on primes in arithmetic progressions. Before describing the result we remark that the conjecture MN(s) was proved by the authors in 2008 ["The Möbius function is strongly orthogonal to nilsequences", preprint, arXiv:0807.1736], and the conjecture GI(s) was recently proved by the authors and the reviewer ["An inverse theorem for the Gowers $U^{s+1}[N]$-norm", preprint, arXiv:1009.3998]; thus the results of the paper under review are no longer conditional.

Consider the following problem: Let $\mathbb{P}$ denote the set of primes. Let A be a $k\times n$ integer matrix, and let $\vec{v}\in\mathbb{Z}^k$ be an integer vector. We ask the following basic question: are there integer-valued vectors $\vec{x}$, so that all coordinates of $A\vec{x}+\vec{v}$ are prime, and if so, how often? In the case where $n=k=1$, the classical Dirichlet theorem asserts that $ax+v$ is prime if and only if there are no "local obstructions", namely $\gcd(a,v)=1$. Furthermore, the Siegel-Walfisz theorem provides the asymptotic number of primes $x\le N$ with $ax+v$ prime.

The authors prove the following theorem: Suppose that no two rows of A are linearly dependent, and that the conjectures MN(s) and GI(s) (to be described below) are true. Then the following local-to-global principle holds: $A\vec{x}+\vec{v}\in\mathbb{P}^k$ infinitely often if and only if for any prime p, there is an integer vector $\vec{x}$ such that all coordinates of $A\vec{x}+\vec{v}$ are in $(\mathbb{Z}/p\mathbb{Z})^*$, and there are infinitely many $\vec{x}$ with $A\vec{x}+\vec{v}$ positive.

Furthermore, the asymptotic number of such integer vectors $\vec{x}$ is given by

$$\left|\left\{\vec{x}\in[-N,N]^n, A\vec{x}+\vec{v}\in\mathbb{P}^k\right\}\right|\sim\mathfrak{S}(A,\vec{v})\frac{N^n}{(\log N)^k},$$

where $\mathfrak{S}(A,\vec{v})$ can be computed explicitly. As a special case one can get, for example, the asymptotic number of k-term arithmetic progressions of prime numbers.

We remark that the condition that no two rows of A are linearly dependent excludes the case of the twin prime conjecture; thus, the paper has no bearing on this problem.

One may also restate the requirement that all coordinates of $A\vec{x}+\vec{v}$ are prime in terms of solving a system of linear equations in prime numbers. Stated in this form, the fewer variables one has, the more difficult the problem is. Thus for example, one equation in two variables, say $x-y=2$, is very difficult. If one allows an additional variable, namely one equation in three variables, then one can solve this equation in primes (assuming no local obstructions) via methods of Vinogradov (1937). Similar methods were used by A. Balog [Mathematika **39** (1992), no. 2, 367–378; MR1203292] for some special cases of systems of linear equations. There was almost no progress on this problem until the breakthrough of the authors [Ann. of Math. (2) **167** (2008), no. 2, 481–547; MR2415379], establishing the existence of arbitrarily long arithmetic progressions in primes. The methods there provided a lower bound of the correct order of magnitude for the number of such progressions, but could not provide asymptotics, nor could they handle nonhomogeneous equations (they rely on E. Szemerédi's theorem [Acta Arith. **27** (1975), 199–245; MR0369312] which does not hold for nonhomogeneous equations and cannot provide asymptotics).

We now give a rough outline of the proof. The authors are counting prime solutions to $\vec{y} = A\vec{x} + \vec{v}$, $A \in M_{k\times n}(\mathbb{Z})$ by considering the solution counting expression

$$\sum_{\vec{x}\in[N]^n} \Lambda(y_1)\cdots\Lambda(y_k),$$

where Λ is the von-Mangoldt function. Studying this function directly is difficult due to the irregular behavior of Λ modulo small primes. Instead they study the same average for the modified functions $\Lambda_{W,b}(n) = \frac{\phi(W)}{W}\Lambda(Wn+b)$, where $w\colon \mathbb{N}\to\mathbb{N}$ is a slow growing function, $W = \prod_{p\le w} p$, and $(W,b) = 1$. The authors show that

$$\frac{1}{N^n}\sum_{\vec{x}\in[N]^n} \Lambda_{W,b_1}(y_1)\cdots\Lambda_{W,b_k}(y_k) = 1 + o(1).$$

When no two rows of A are linearly dependent, the average above is "controlled" by the so-called Gowers uniformity norms, introduced by W. T. Gowers in [Geom. Funct. Anal. **11** (2001), no. 3, 465–588; MR1844079], in the sense that the estimate above holds if $\|\Lambda_{W,b} - 1\|_{U^s} = o(1)$ for some integer s depending on A. This is where GI(s) kicks in.

The Inverse Theorem for the Gowers norms U^s (GI(s)) [B. Green, T. Tao and T. Ziegler, op. cit.] states that a bounded function $f\colon [N]\to\mathbb{C}$ satisfies $\|f\|_{U^s} = o(1)$ if and only if f does not correlate with bounded $(s-1)$-step nilsequences, namely

$$\sum_{n\le N} f(n)g(n) = o(N)$$

for any bounded function $g\colon \mathbb{Z}\to\mathbb{C}$ of the form $g(n) = F(a^n x)$ where $F\colon G/\Gamma\to\mathbb{C}$ is a Lipschitz continuous function, G/Γ is an $(s-1)$-step nilmanifold, and $a\in G$. Using the transference principle introduced by the authors in [Ann. of Math. (2) **167** (2008), no. 2, 481–547; MR2415379], one can transfer this criterion to functions bounded by pseudorandom measures, thus it applies to the (non-bounded) function $\Lambda_{W,b} - 1$. It follows that it suffices to check that $\Lambda_{W,b} - 1$ does not correlate with an $(s-1)$-step nilsequence; namely

$$\sum_{n\le N} (\Lambda_{W,b} - 1)(n)g(n) = o(N),$$

where g is as described above. This condition in turn can be translated to a similar condition replacing $\Lambda_{W,b} - 1$ by the Möbius function $\mu(n)$; but this is the statement of the Möbius Nilsequence Theorem (MN(s)) [B. Green and T. Tao, op. cit., arXiv:0807.1736].

Tamar Ziegler
From MathSciNet, October 2016

MR2912706 11B75; 05D10, 37A45

Polymath, D. H. J.

A new proof of the density Hales-Jewett theorem.

Annals of Mathematics. Second Series **175** (2012), *no.* 3, 1283–1327.

The following standard notation is used: For any integers $k \geq 1$ and $n \geq 0$, $[k] := \{1, \ldots, k\}$ and $[k]^n$ is the set of all words of length n on the alphabet $[k]$. If w is a word on the alphabet $[k] \cup \{x\}$ in which at least one x appears (x is sometimes called a wildcard letter), then for $i \in [k]$, $w(i)$ denotes the word obtained from w by replacing each x by i. A combinatorial line is a set of the form $\{w(i) : i = 1, \ldots, k\}$.

The density Hales-Jewett (DHJ) theorem states that for all k and $\epsilon > 0$, there exists n such that if A is any subset of $[k]^n$ with $|A| > \epsilon k^n$, then A contains a combinatorial line.

Here is a long quotation from the paper under review, which gives the motivation for finding a new proof:

"Why is it interesting to give a new proof of the density Hales-Jewett theorem?" There are two main reasons. The first is connected with the history of results and techniques in this area. One of the main benefits of Furstenberg's proof of Szemerédi's theorem was that it introduced a technique—ergodic methods—that could be developed in many directions, which did not seem to be the case with Szemerédi's proof. As a result, several far-reaching generalizations of Szemerédi's theorem were proved [V. Bergelson and A. Leibman, J. Amer. Math. Soc. **9** (1996), no. 3, 725–753; MR1325795; H. Furstenberg and Y. Katznelson, J. Analyse Math. **34** (1978), 275–291 (1979); MR0531279; J. Analyse Math. **45** (1985), 117–168; MR0833409; J. Anal. Math. **57** (1991), 64–119; MR1191743], and for a long time nobody could prove them in any other way than by using Furstenberg's methods. In the last few years that has changed, and a programme has developed to find new and finitary proofs of the results that were previously known only by infinitary ergodic methods; see, e.g., [V. Rödl and J. Skokan, Random Structures Algorithms **25** (2004), no. 1, 1–42; MR2069663; B. Nagle, V. Rödl and M. Schacht, Random Structures Algorithms **28** (2006), no. 2, 113–179; MR2198495; V. Rödl and J. Skokan, Random Structures Algorithms **28** (2006), no. 2, 180–194; MR2198496; V. Rödl and M. Schacht, Combin. Probab. Comput. **16** (2007), no. 6, 833–885; MR2351688; Combin. Probab. Comput. **16** (2007), no. 6, 887–901; MR2351689; W. T. Gowers, Combin. Probab. Comput. **15** (2006), no. 1-2, 143–184; MR2195580; Ann. of Math. (2) **166** (2007), no. 3, 897–946; MR2373376; T. C. Tao, Electron. J. Combin. **13** (2006), no. 1, Research Paper 99, 49 pp.; MR2274314; J. Anal. Math. **103** (2007), 1–45; MR2373263]. Giving a nonergodic proof of the density Hales-Jewett theorem was seen as a key goal for this programme, especially since Furstenberg and Katznelson's ergodic proof was significantly harder than the ergodic proof of Szemerédi's theorem. Having given a purely finitary proof, we are able to obtain explicit bounds for how large n needs to be as a function of δ and k in the density

Hales-Jewett theorem. Such bounds could not be obtained via the ergodic methods even in principle, because these proofs rely on the Axiom of Choice. Admittedly, our explicit bounds are not particularly good: we start with a tower-type dependence for $k = 3$ and go up a level of the Ackermann hierarchy each time we go from k to $k + 1$. However, they are in line with several other bounds in the area. For example, the best known bounds for the multidimensional Szemerédi theorem [W. T. Gowers, op. cit.; MR2373376; B. Nagle, V. Rödl and M. Schacht, op. cit.] (which is an easy consequence of DHJ) are also of this type.

"A second reason that a new proof of the density Hales-Jewett theorem is interesting is that it immediately implies Szemerédi's theorem, and finding a new proof of Szemerédi's theorem seems always to be illuminating—or at least this has been the case for the four main approaches discovered so far (combinatorial [E. Szemerédi, Acta Arith. **27** (1975), 199–245; MR0369312], ergodic [H. Furstenberg, J. Analyse Math. **31** (1977), 204–256; MR0498471; H. Furstenberg, Y. Katznelson and D. S. Ornstein, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 527–552; MR0670131], Fourier [W. T. Gowers, Geom. Funct. Anal. **11** (2001), no. 3, 465–588; MR1844079], hypergraph removal [W. T. Gowers, op. cit.; MR2195580; op. cit.; MR2373376; V. Rödl and J. Skokan, op. cit.; MR2069663; B. Nagle, V. Rödl and M. Schacht, op. cit.]). Surprisingly, in view of the fact that DHJ is considerably more general than Szemerédi's theorem and the ergodic-theory proof of DHJ is considerably more complicated than the ergodic-theory proof of Szemerédi's theorem, the new proof we have discovered gives arguably the simplest proof yet known of Szemerédi's theorem. It seems that by looking at a more general problem we have removed some of the difficulty. Related to this is another surprise. We started out by trying to prove the first difficult case of the theorem, DHJ_3 [the case $k = 3$]. The experience of all four of the earlier proofs of Szemerédi's theorem has been that interesting ideas are needed to prove results about progressions of length 3, but significant extra difficulties arise when one tries to generalize an argument from the length-3 case to the general case. Unexpectedly, it turned out that once we had proved the case $k = 3$ of the density Hales-Jewett theorem, it was straightforward to generalize the argument to the $k \geq 4$ cases. We do not fully understand why our proof should be different in this respect, but it is perhaps a sign that the density Hales-Jewett theorem is at a 'natural level of generality'."

A paragraph further on are some crucial remarks regarding the origin of the pseudonym "D. H. J. Polymath". "Polymath" says:

"Before we start working towards the proof of the theorem, we would like briefly to mention that it was proved in a rather unusual 'open source' way, which is why it is being published under a pseudonym. The work was carried out by several researchers, who wrote their thoughts, as they had them, in the form of blog comments at `http://gowers.wordpress.com`. Anybody who wanted to could participate, and at all stages of the process the comments were fully open to anybody who was interested. ... The blog comments are still available, so although this paper is a polished account of the DHJ_k argument, it is possible to read a record of the entire thought process that led to the proof. ... The participants in the project also created a wiki, `michaelnielsen.org/polymath1`, which contains sketches of the arguments, links to the blog comments, and a great deal of related material."

This reviewer would have preferred to see, rather than the pseudonym "Polymath", a list of authors. In other fields there are papers with a hundred co-authors.

Why not in mathematics a paper with twenty or thirty co-authors, with extra credit for the person(s) who wrote the exposition?

In any case, the blog comments make interesting reading, and one can see there (given enough effort!) names associated with some of the major and minor steps in the proof.

When reading this extraordinary paper, it would no doubt be helpful to have at hand the clear outline of the proof in Gowers' paper [in *An irregular mind*, 659–687, Bolyai Soc. Math. Stud., 21, János Bolyai Math. Soc., Budapest, 2010; MR2815619].

T. C. Brown
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Tao, Terence

Higher order Fourier analysis. (English)

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American Mathematical Society, *Providence, RI*, 2012, x+187 pp.,
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Recent progress in additive combinatorics has shown the need for higher-order analogues of Fourier analysis. Where classical Fourier analysis considers functions in terms of linear phase functions such as $n \mapsto e(\alpha n)$, higher-order Fourier analysis also allows quadratic and higher-order phase functions such as $n \mapsto e(\alpha n^2)$. The need for analysing such functions was first raised in the seminal work of W. T. Gowers [Geom. Funct. Anal. **8** (1998), no. 3, 529–551; MR1631259; Geom. Funct. Anal. **11** (2001), no. 3, 465–588; MR1844079] on Szemerédi's theorem and also arose naturally in parallel, but related, work in ergodic theory [B. Host and B. Kra, Ann. of Math. (2) **161** (2005), no. 1, 397–488; MR2150389]. More recently, it has played a key role in the work of B. Green, Tao and T. D. Ziegler on linear equations in the primes [B. Green and T. C. Tao, Ann. of Math. (2) **171** (2010), no. 3, 1753–1850; MR2680398; Ann. of Math. (2) **175** (2012), no. 2, 541–566; MR2877066; B. Green, T. C. Tao and T. D. Ziegler, Ann. of Math. (2) **176** (2012), no. 2, 1231–1372; MR2950773]. This book serves as an introduction to this nascent field.

The book is split into two parts. The first part is the core of the book and discusses the origins and applications of higher-order Fourier analysis, building towards a discussion of the author's work, with Green and Ziegler, on asymptotics for linear equations in the primes. Along the way, he discusses a number of topics of interest, including the classical theory of equidistribution (from multiple perspectives), Roth's theorem on three-term arithmetic progressions in dense subsets of the integers (again from multiple perspectives), and the inverse theorems for the Gowers uniformity norms. These inverse theorems play a key role both in proving Szemerédi's theorem and in deriving the correct asymptotics for linear equations in the primes, and their study has been one of the key factors behind the development of a higher-order Fourier analysis.

The second part of the book consists of edited versions of a number of related posts taken from Tao's blog. There is a lengthy discussion of ultralimit analysis (which was used in the work of Green, Tao and Ziegler on inverse theorems for the uniformity norms) and its applications to quantitative algebraic geometry, in particular a quantitative version of a theorem of Gromov on groups of polynomial growth. The author also discusses higher-order analogues of Hilbert spaces, where the usual binary inner product is replaced by a 2^d-ary inner product between 2^d functions, and the classical uncertainty principle.

David Conlon

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