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Lecture notes on mean curvature flow: barriers and singular perturbations, by Giovanni Bellettini, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie), Vol. 12, Edizioni della Normale, Pisa, 2013, xviii+325 pp., ISBN 978-88-7642-428-1, US$29.99; electronic US$19.99

The mean curvature flow is a nonlinear geometric evolution equation where a submanifold evolves over time to minimize its volume. In the evolution, each point moves in the inward normal direction with speed given by the mean curvature. Thus convex points move inwards, concave points move outwards, and the submanifold moves faster where the curvature is larger. The submanifold is static—or constant in time—if the mean curvature vanishes, i.e., if it is minimal submanifold.

Mean curvature flow originated in the materials science literature, starting in the 1920s, with the current formulation being given by Mullins in the 1950s [M]. It has been studied extensively in applied mathematics, image processing, and other areas of science and engineering. Of particular interest here, mean curvature flow arises in the study of the asymptotic behavior as $\epsilon \to 0$ of solutions of the Allen–Cahn equation

$$\epsilon \frac{\partial u_\epsilon}{\partial t} = \epsilon \, \Delta u_\epsilon - \frac{1}{2\epsilon} \, u_\epsilon(u_\epsilon^2 - 1) \tag{1}$$

used to model phase transitions. In pure mathematics, a discrete version of the flow, known as *Birkhoff's curve shortening process*, was used by Birkhoff in the 1910s to prove that every closed Riemannian surface contains a closed geodesic. The continuous flow has been studied systematically starting with the pioneering book of Brakke in 1978 [B].

Mean curvature flow has been studied for submanifolds of any codimension in Riemannian manifolds of any dimension. We will restrict discussion to the evolution of hypersurfaces in Euclidean space.

The simplest example of mean curvature flow is the evolution of round n-spheres, where the symmetry is preserved and, thus, it reduces to an ODE for the radius $R(t)$

$$R'(t) = -\frac{n}{R(t)} \, .$$

Integrating this ODE gives $R(t) = \sqrt{R_0 - 2nt}$ where R_0 is the radius at time $t = 0$. Observe that the radius always goes to zero in finite time and then the solution disappears. This is known as *finite time extinction*. It is a singularity of the flow—the spheres are varying smoothly as hypersurfaces but then just disappear at the extinction time.

The parabolic maximum principle leads to an avoidance principle for the flow: if two smooth compact flows are initially disjoint, then they remain disjoint. It follows that the flow starting from any smooth closed hypersurface must become singular before a sphere enclosing it becomes extinct. Thus, singularities are unavoidable. A major focus of research in mean curvature flow is to understand these singularities: What do they look like? How many are there? Can the flow be continued past them? Which ones can be perturbed away?

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When the submanifold is a curve, the flow is known as the curve shortening flow. A remarkable result of Grayson from 1987 (using earlier work of Gage and Hamilton) shows that any simple closed curve in the plane remains smooth under the flow until it disappears in finite time in a point. Right before it disappears, the curve will be an almost round circle. More precisely, if we take a sequence of times going to the exinction time and rescale the curves at each of these times to contain area π, then the sequence of rescaled curves will converge (smoothly) to the round unit circle. Thus, each flow has only one singularity in all of space and time and this singularity looks just like the shrinking circles.

In higher dimensions, Huisken showed in 1984 that closed convex hypersurfaces remain convex and flow smoothly up until they become extinct at a point and, moreover, they are almost round just before extinction. However, unlike the case of curves, there are many new types of singularities when the initial hypersurface is not convex. The main tool for analyzing these singularities is a blow-up method, similar to tangent cone analysis for minimal varieties, that relies on a monotonicity formula of Huisken from 1990 [H], [W]. The flow once again has a self-similar structure near the singularities, but there are infinite families of different possible self-similar structures. Recent years have seen a great deal of activity studying this, constructing examples, classifying the possibilities in certain cases, and understanding which types of singularities are generic; see the survey [CMP] and references therein.

There are several different ways to approach mean curvature flow, each with its own advantages and disadvantages. Perhaps the most natural is to write the evolving hypersurface as the image of an evolving map from a fixed submanifold (e.g., the initial hypersurface). This is useful for many problems, but one major drawback is that it is only meaningful before the first singular time. Brakke's geometric measure theory approach studies the evolution of weak, measure-theoretic objects known as varifolds; these deal well with singularities, but other issues become more complicated. The level set method views the hypersurface as a level set of an evolving function, allowing for singularities when the level set contains a critical point. This has been done with great success numerically by Osher and Sethian [OS], with theoretical existence and uniqueness established by Evans and Spruck and Chen, Giga, and Goto in the early 1990s; see the book [G]. There are a number of other approaches, including time discretization, set-theoretic weak solutions, and others.

This book by Bellettini takes a different approach to the flows. The evolving surface is viewed as the boundary $\partial f(t)$ of an evolving solid subset $f(t)$. The mean curvature flow equation is then expressed as a differential equation for the signed distance function $d(x,t)$ to $\partial f(t)$. To explain this, recall that the signed distance is given by

$$d(x,t) = \mathrm{dist}(x, f(t)) - \mathrm{dist}(x, \mathbf{R}^n \setminus f(t)) \, ,$$

so that $d(x,t)$ is positive outside of $f(t)$, zero on $\partial f(t)$, and negative inside of $f(t)$. The gradient of d is a unit vector (away from cut points). The Hessian of d at points on $\partial f(t)$ is just the second fundamental form of the boundary $\partial f(t)$ and, thus, the mean curvature is the Laplacian Δd. Using this, the mean curvature flow equation can be expressed as

$$\frac{\partial d}{\partial t} = \Delta d \text{ along } \partial f(t) \, . \tag{2}$$

For example, when $f(t)$ is the solid n-ball of radius $R(t) = \sqrt{R_0 - 2nt}$ in $\mathbf{R}^{n+1}$, then

$$d(x,t) = |x| - R(t)\,.$$

Since $\Delta|x| = \frac{n}{|x|}$ in $\mathbf{R}^{n+1}$ and $R'(t) = -\frac{n}{R(t)}$, we see that $d(x,t) = |x| - R(t)$ satisfies (2).

The equation (2) can be used to prove many of the properties of the flow, such as the avoidance principle, but for many other things it is more useful to have an equation not just where d vanishes. This equation can be modified to extend to a tubular neighborhood of the evolving boundary, though the right-hand side becomes nonlinear in the Hessian Hess_d of d

$$\frac{\partial d}{\partial t} = \mathbf{Trace}\left(\text{Hess}_d\,(\delta_{ij} - d\,\text{Hess}_d)^{-1}\right)\,, \tag{3}$$

where δ_{ij} is the identity matrix. When $d = 0$, the right-hand side becomes the trace of the Hessian, which is just the Laplacian, and we recover the previous equation.

This formulation of the mean curvature flow is well suited for a theory of weak solutions since level sets of functions need not be smooth. The one developed here is the theory of barriers. A barrier is a one-parameter family of sets that serves as a supersolution of mean curvature flow. Roughly, this means that it does not violate the maximum principle when compared to any smooth mean curvature flow. This originates in a series of papers by De Giorgi in the early 1990s and the presentation here follows developments by Bellettini and collaborators over the last 20 years. The intersection of two barriers is shown to also be a (smaller) barrier and the weak solution is then given by the minimal barrier. This is shown to coincide with the smooth solution (when it exists) and to satisfy the same avoidance principle as smooth solutions: two initially disjoint minimal barriers remain disjoint and, in fact, the distance between them grows.[1]

The last three chapters of the book are devoted to the asymptotics of the Allen–Cahn equation (1), including its variational formulation, the formal asymptotics in ϵ, and its relationship to mean curvature flow. The main result is that the level sets of solutions to the Allen–Cahn equation converge, under mild hypotheses, to solutions of mean curvature flow. This remains a very active and rapidly developing area in both pure and applied mathematics, and many open questions remain.

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