William Feller, Selected papers I, René L. Schilling, Zoran Vondracek, and Wojbor
A. Woyczyński (Editors), Springer International Publishing, Cham, Switzerland,
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William Feller (1906–1970) made outstanding contributions to the limit theory
of sums of random variables (central limit problem, law of large numbers, and law
of the iterated logarithm), recurrent event and discrete renewal theory, Markov
processes, the application of regular variation in probability, and much else. The
first volume of his *Introduction to probability theory and its applications, Volume I,*
[John Wiley & Sons, Inc., New York–London–Sydney, 1950] is described by Gian-
Carlo Rota (dust jacket of the third edition and quoted in *Ann. Math. Statist.* **41**
(1970), p. viii)) as:

> one of the great events in mathematics of this century. Together
> with Weber's *Algebra* and Artin's *Geometric Algebra* this is the
> finest textbook in mathematics in this century. It is a delight to
> read and it will be immensely useful to scientists in all fields.

Volume II [John Wiley & Sons, Inc., New York–London–Sydney, 1966] is no less
useful and has been described (S. Orey, [MR0210154]) as having:

> a rich texture, derived from the wealth of problems treated as appli-
> cations or illustrations of the theory. The striking aspect is the ap-
> parent ease and elegance with which these problems are dispatched,
> frequently making obsolete the original treatment given in the re-
> search literature.

In 1986, while on a sabbatical visit, I was hosted by a senior probabilist, and
our conversation came round to Feller and his work. I said it was about time
someone assembled Feller's probability papers. My host disagreed, saying that
Feller's probability texts comprised a matchless tribute to him. Now, thirty years
later, the probability community will welcome these two volumes of selected papers
as well as their additional material.

The front matter of each volume contains Feller's CV, his bibliography, and a
listing of his PhD students. Volume I proper begins with three published tributes
written by Z. W. Birnbaum, J. L. Doob, and M. Kac, respectively. A substantial
biographical essay by Schilling and Woyczyński is printed in each volume. It ap-
pears to be based on a richly illustrated online biography written by D. Zubrinic
(`www.croatianhistory.net/etf/feller.html`).

Volume I continues with three essays (about 60 pages): "Feller's early work on
measure theory and mathematical foundations of probability" and "Feller's early
work on limit theorems" by H. Fischer, and "Feller and Busemann on surface
theory—Contributions to geometry" by E. Scholz.

The essays are followed by a selection of papers covering the period from 1928 to 1949. Feller's bibliography comprises 109 items which include 103 papers, 5 monographs (editions and volumes of *Introduction*), and one review paper translated into Spanish. Sixty-nine papers have been chosen for inclusion in the two volumes. Feller wrote in German until 1939, the year he migrated to the United States. The most significant of his German language papers in Volume I are accompanied by translations into English by René Schilling. These translations include footnotes indicating corrections to the originals and explanatory remarks. The symbol "¶" is used to indicate page breaks in the German originals.

The English translations include all of Feller's research papers on limit theory and Markov processes and one on von Mises *kollectivs*. Here are the titles of these English translations and English language selections in the volume:

- "On the central limit theorem of probability theory"
- "On the theory of stochastic processes. (Existence and uniqueness)"
- "On the central limit theorem of probability theory II"
- "On the law of large numbers"
- "The foundations of Volterra's theory of the struggle for life in a probabilistic treatment"
- "Completely monotone functions and sequences"
- "On the existence of so-called kollektivs"
- "On the integro-differential equations of purely discontinuous Markoff processes"
- "On the integral equation of renewal theory"
- "Some geometric inequalities"
- "Generalization of a probability limit theorem of Cramér"
- "The general form of the so-called law of the iterated logarithm"
- "On a general class of 'contagious' distributions"
- "On the normal approximation to the binomial distribution"
- "The fundamental limit theorems in probability"
- (With H. Busemann) "Regularity properties of a certain class of surfaces"
- "Note on the law of large numbers and 'fair' games"
- "A limit theorem for random variables with infinite moments"
- "The law of the iterated logarithm for identically distributed random variables"
- "On the Kolmogorov–Smirnov limit theorems for empirical distributions"
- "On probability problems in the theory of counters"
- (With P. Erdős and H. Pollard) "A property of power series with positive coefficients"
- (With K. L. Chung) "On fluctuations in coin tossing"
- "On the theory of stochastic processes, with particular reference to applications"
- "Fluctuation theory of recurrent events"

Feller published no papers during 1950 when, presumably, he was finalising the writing and publication of Volume I of his *Introduction*.

Selected Papers II begins with the following essays (about 100 pages): "Feller's contributions to mathematical biology" by E. Baake and A. Wakolbinger, "Feller on differential operators and semi-groups" by N. Jacob, "Feller's contributions to the one-dimensional diffusion theory and beyond" by M. Fukushima, "On boundary

behaviour of one-dimensional diffusions: From Brown to Feller and beyond" by G. Peskir, and "Feller's work in renewal theory, the law of the iterated logarithm and Karamata theory" by R. Maller.

Selected Papers II includes papers published from 1951 to 1970 on the semigroup treatment of Markov processes, regular variation, further work on limit and renewal theory, and other topics, as follows:

- "The asymptotic distribution of the range of sums of independent random variables"
- "Diffusion processes in genetics"
- "Two singular diffusion problems"
- "The parabolic differential equations and the associated semigroups of transformations"
- "On a generalization of Marcel Riesz's potentials and the semigroups generated by them"
- "Some recent trends in the mathematical theory of diffusion"
- "On positivity preserving semigroups of transformations on $C[r_1, r_2]$"
- "Semigroups of transformations in general weak topologies"
- "On the generation of unbounded semigroups of bounded linear operators"
- "The general diffusion operator and positivity preserving semigroups in one dimension"
- "Diffusion processes in one dimension"
- "On second order differential operators"
- "On differential operators and boundary conditions"
- "Boundaries induced by non-negative matrices"
- (With H. P. McKean Jr.) "A diffusion equivalent to a countable Markov chain"
- (With J. Elliott) "Stochastic processes connected with harmonic functions"
- "Generalized second order differential operators and their lateral conditions"
- "On boundaries and lateral conditions for the Kolmogorov differential equations"
- "On boundaries defined by stochastic matrices"
- "On the intrinsic form for second order differential operators"
- "The birth and death processes as diffusion processes"
- "Non-Markovian processes with the semigroup property"
- "Differential operators with the positive maximum property"
- "Some new connections between probability and classical analysis"
- (With S. Orey) "A renewal theorem"
- "A simple proof for renewal theorems"
- "On the classical Tauberian theorems"
- "On semi-Markov processes"
- "On the Fourier representation for Markov chains and the strong ratio theorem"
- "On the influence of natural selection on population size"
- "Infinitely divisible distributions and Bessel functions associated with random walks"
- "On regular variation and local limit theorems"
- "On fitness and the cost of natural selection"

- "On the Berry–Esseen theorem"
- "An extension of the law of the iterated logarithm to variables without variance"
- "One-sided analogues of Karamata's regular variation"
- "General analogues to the law of the iterated logarithm"
- "Limit theorems for probabilities of large deviations"
- "On the oscillations of sums of independent random variables"

Generally speaking, the essays in each volume review Feller's contributions and discuss them in the context of subsequent research. Does it make any sense to ask for Feller's most important contribution? There is so much to choose from! G. Peskir writes in his above essay:

> Feller's greatest discovery in mathematics was *sticky* (or *slowly varying*) boundary behaviour of one-dimensional diffusion processes (characterised by the appearance of the *second derivative*) at the boundary point.

In *Indiscrete Thoughts* [Birkhäuser, Boston (1997) [MR1419503]] G.-C. Rota, while writing about Volume I of the *Introduction*, says about Chapter XIII,

> The treatment of recurrent events was the one he rewrote most, and it is still, strictly speaking, wrong. Nevertheless, it is perhaps his greatest piece of work. We are by now so used to Feller's ideas that we tend to forget how much mathematics today goes back to his "recurrent events"; the theory of formal grammars is one outlandish example.

There is a great deal on offer in these two excellent volumes, and they deserve a place in any library collection devoted to probability!

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