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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
LEVINE AND PERES

MR2246566 (2007g:82042) 82C27, 82-02, 82B20, 82C03, 82C20, 82C41

Dhar, Deepak

Theoretical studies of self-organized criticality. (English summary)

Physica A. Statistical Mechanics and Its Applications **369** (2006), *no.* 1, 29–70.

This review paper gives an introduction to the abelian sandpile and related models of self-organized criticality (SOC), from the point of view of theoretical physics. It is based on lectures Dhar gave at the École Polytechnique Fédérale Lausanne in 1998, and has been reorganized and substantially updated compared to the earlier electronic version [D. Dhar, "Studying self-organized criticality with exactly solved models", preprint, arXiv.org/abs/cond-mat/9909009]. The notes aim to provide a pedagogical introduction to SOC for students and others wishing to learn about the subject in detail. It will be useful reading to anyone interested in theoretical results on SOC, since it summarizes some of the main results in the field, reflecting choices of the author. A lot of information is condensed into about 40 pages, and an extensive list of references to original papers is given where the interested reader can find more information. The exposition is that of a physics paper, nevertheless, researchers interested in mathematical results will find that this usually only amounts to a change in style, since many of the results are mathematically rigorous. In summary, there is a wealth of useful background, and interesting open questions.

The following topics are discussed, among others:

1. models of SOC: the BTW (abelian) sandpile; loop-erased random walk; Takayasu's model.
2. mathematical structures in the sandpile model, such as: abelian group of addition operators; the steady state; representation of the sandpile group as a product of cyclic groups; the two-point function; recurrent and transient configurations; the burning test.
3. relation to the $q \to 0$ limit of the Potts model and spanning trees; decomposition of avalanches into waves.
4. directed sandpiles: computation of critical exponents in all dimensions; equivalence to other models: river networks, the voter model.
5. summary of results for undirected sandpiles on $\mathbb{Z}^d$ and on the Bethe lattice.
6. more general abelian models: Eulerian walkers; the Manna model.
7. time-dependent properties.
8. conjectures regarding generic behaviour and universality: the effects of stochasticity in toppling rules and 'stickyness'.
9. open problems.

Antal A. Járai

From MathSciNet, April 2017

MR1188055 (94a:60105) 60J15; 60K35, 82B41

Lawler, Gregory F.; Bramson, Maury; Griffeath, David

Internal diffusion limited aggregation. (English summary)

The Annals of Probability **20** (1992), *no.* 4, 2117–2140.

The authors consider the following stochastic growth model, which is a variant of the standard DLA model of T. A. Witten, Jr. and L. M. Sander [Phys. Rev. Lett. **47** (1981), no. 19, 1400–1403], and which was suggested by P. Diaconis and W. Fulton [Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), no. 1, 95–119 (1993)]. $A_1 = \{0\}$ and A_n is a connected set of n lattice sites on $\mathbf{Z}^d$ containing the origin. A_{n+1} is formed from A_n by adding point y_n from ∂A_n to A_n. Here ∂A_n is the set of points adjacent to A_n, but not in A_n. To choose y_n one starts a simple random walk at the origin and lets it run until it visits ∂A_n for the first time. This first hitting point of ∂A_n by the random walk is taken for y_n. (For later purposes we also introduce σ_n as the time at which this random walk first hits ∂A_n.) One then starts the procedure anew to find y_{n+1}, etc. The only difference with the Witten-Sander model is that here the random walks start at the origin, which lies inside A_n, while Witten and Sander start the random walk "at infinity", far outside A_n. This makes a great difference to the shape of the aggregates A_n. In the Witten-Sander version it is believed (but not yet proven) that A_n grows long and spindly arms. Here the authors prove that, for their "internal DLA", A_n is asymptotically spherical, i.e., $n^{-1/d}A_n$ converges with probability 1 to a Euclidean ball with a fixed radius $C(d)$. The authors also find the probabilistic behavior of the total number of random walk steps which are needed for the formation of A_n, that is $\sigma_1 + \cdots + \sigma_n$, and also consider a continuous-time variant.

H. Kesten

From MathSciNet, April 2017

MR1962779 (2003m:60285) 60K37; 60F10, 60G70, 60K35

Ben Arous, Gérard; Quastel, Jeremy; Ramírez, Alejandro F.

Internal DLA in a random environment. (English, French summaries)

Annales de l'Institut Henri Poincaré. Probabilités et Statistiques **39** (2003), *no.* 2, 301–324.

Internal DLA is a stochastic particle system in which independent traps are distributed on a d-dimensional integer lattice, and particles are produced at the origin and move as independent random walk until they hit a trap, at which time the particle stops, and the trap becomes saturated. The aim of this paper is to present the asymptotic shape and growth rate for the set of saturated traps. This study follows previous work from the authors on this subject. Three different regimes appear according to the behavior of the number of particles injected up to time t, $N(t)$.

The subcritical case corresponds to $\lim_{t\to\infty} t^{-d/2}N(t) = 0$. Then the asymptotic shape is a ball in any dimension with radius proportional to $N(t)/m$, where m is the probability to observe a trap at a given site. In this case there is asymptotically a zero density of live particles. This result was already proved by two of the authors.

The critical case corresponds to $\lim_{t\to\infty} t^{-d/2}N(t) \in (0,\infty)$. Then, in any dimension, the asymptotic shape is a ball with radius proportional to $\sqrt{t}$ times a constant which depends on m. There is a nontrivial density of live particles. The boundary of the asymptotic shape and the density of particles evolve according to a one-phase Stefan problem. The proof relies on hydrodynamic limit methods.

The supercritical case corresponds to $\lim_{t\to\infty} t^{-d/2}N(t) \in (0,\infty)$. It is studied in dimension one using tail estimates on order statistics of independent random walks. It exhibits two different behaviors according to subexponential or superexponential behavior of $N(t)$. In the subexponential case, the traps slow down the growth rate in a way which does not depend on m. In the superexponential case, the effect of traps disappears, and the speed of propagation is controlled by the range of free random walks.

Jean-François Delmas
From MathSciNet, April 2017

MR2824564 (2012k:60137) 60G50; 31C20, 39A45, 52C20, 60J67
Chelkak, Dmitry; Smirnov, Stanislav
Discrete complex analysis on isoradial graphs. (English summary)
Advances in Mathematics **228** (2011), *no.* 3, 1590–1630.

This is the first in a series of papers dedicated to establishing universality of the critical Ising model with respect to a large class of graphs with neighboring vertices all equidistant from common centers. Applications of the results in this paper extend beyond the Ising model; universality of the 2d loop-erased random walk and natural approximations of harmonic and holomorphic functions on continuous subdomains of $\mathbb{C}$ also follow.

Section 2 and the Appendix provide a dense survey of definitions and results concerning discrete harmonic functions and discrete holomorphic functions defined on 2d isoradial graphs; square lattices represent a special case of isoradial graphs with a history of intensive research. A comprehensive survey of established definitions and results in this field is included in Section 2, especially drawing from R. J. Duffin's ideas [Duke Math. J. **20** (1953), 233–251; MR0070031] and recent work of R. W. Kenyon [Invent. Math. **150** (2002), no. 2, 409–439; MR1933589] and C. Mercat ["Discrete polynomials and discrete holomorphic approximation", preprint, `arXiv:math-ph/0206041`].

Section 2 allows the authors to establish consistent notation for the main results, but most calculations are omitted. An attentive reader must depend on a firm grasp of complex analysis and a keen awareness of the geometry of underlying rhombic lattices (isoradial graphs form rhombic tessellations when combined with their dual). The figures included are extremely helpful, and reference copies of Figures 1 (A) and (B) are useful while reading Section 2. Readers unfamiliar with the theory may get confused by some errors in the definitions; the latter equality in the definition of $\mathrm{Proj}[F;\psi]$ appears to be incorrect, and the vocabulary used to define prime ends in Subsection 3.2 is self-referential.

Section 3 is dedicated to establishing limits with respect to mesh width, δ, of families of pairs of holomorphic or harmonic functions with the subgraphs (of radius δ) they are defined on. The definitions of these subgraphs, limits of these subgraphs and the topology under which convergence is established require a series of definitions, and statements of the propositions are often as long as the proofs themselves. The reader is aided by recalling parallel results established in the context of precompact sequences of analytic functions in the classical context. The final subsection requires additional technicalities in order to establish convergence of discrete Poisson kernels to continuous counterparts on a more general class of discrete domains, a result particularly important in establishing convergence of interfaces arising from the critical models on isoradial graphs to Schramm-Loewner evolution.

Ben Dyhr
From MathSciNet, April 2017

MR2449057 (2009h:91004) 91A05; 35J60, 35J70, 49N70, 91A23
Peres, Yuval; Schramm, Oded; Sheffield, Scott; Wilson, David B.
Tug-of-war and the infinity Laplacian.
Journal of the American Mathematical Society **22** (2009), *no.* 1, 167–210.

In this very interesting paper, the authors consider a two-person differential game called "tug-of-war". Here is the rough description of the game when it is on a bounded open set $\Omega \subset \mathbb{R}^n$. For fixed $\epsilon > 0$, starting from $x_0 \in \Omega$, at the kth turn, the two players toss a coin and the winner chooses an $x_k \in \Omega$ with $|x_k - x_{k-1}| \leq \epsilon$. The game ends when $x_k \in \partial\Omega$. Player I tries to maximize its payoff

$$F(x_k) + \frac{\epsilon^2}{2}\sum_{i=0}^{k-1} f(x_i)$$

and player II tries to minimize it. Here $F \in C(\partial\Omega)$ is the terminal payoff function and $f \in C(\Omega)$ is the running payoff function. Let u_ϵ be the value of the above game. If $\inf_{\overline{\Omega}} f > 0$ or $f \equiv 0$, the authors show that

$$\lim_{\epsilon \to 0} u_\epsilon = u,$$

where u is the unique viscosity solution of the following equation:

$$\begin{cases} \dfrac{\Delta_\infty u}{|Du|^2} = -f & \text{in } \Omega \\ u|_{\partial\Omega} = F, \end{cases}$$

where $\Delta_\infty u = \sum_{i=1}^n u_{x_i} u_{x_j} u_{x_i x_j}$.

When $f \equiv 0$, the above equation is the famous infinity Laplacian equation which was firstly introduced by [G. Aronsson, Ark. Mat. **6** (1967), 551–561 (1967); MR0217665]. It is the Euler-Lagrange equation of minimizing the Lipschitz constant. Precisely speaking, we say that $u : \overline{\Omega} \to \mathbb{R}$ is a minimal Lipschitz extension of $F : \partial\Omega \to \mathbb{R}$ if

$$\mathrm{Lip}_{\overline{\Omega}} u = \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{\|x - y\|} = \sup_{x,y \in \partial\Omega} \frac{|F(x) - F(y)|}{\|x - y\|} = \mathrm{Lip}_{\partial\Omega} F,$$

and we say u is an absolute minimal Lipschitz extension of F if additionally for any open set $V \subset \Omega \backslash \partial\Omega$ one has,

$$\sup_{x \neq y \in \overline{V}} \frac{|u(x) - u(y)|}{|x - y|} = \sup_{x \neq y \in \partial V} \frac{|u(x) - u(y)|}{|x - y|}.$$

It was proved by [R. R. Jensen, Arch. Rational Mech. Anal. **123** (1993), no. 1, 51–74; MR1218686] that u is a minimal absolute Lipschitz extension of F if and only if it is a viscosity solution of the infinity Laplacian equation (i.e., with $f \equiv 0$) with boundary value F. Jensen also showed that the Dirichlet problem of the infinity Laplacian equation has a unique solution.

This article provides a game theory point of view for understanding the infinity Laplacian equation and a new proof of Jensen's influential uniqueness result based on probabilistic approaches (actually more than that when $f > 0$). So far, our understanding of the infinity Laplacian equation is quite limited. Insights and intuitions from game theory might help make this extremely interesting but mysterious equation more understandable. Some progress has already been made in this paper.

Moreover, the "tug-of-war" game gives a meaning to the normalized infinity Laplacian operator $\tilde{\Delta}_\infty u = \frac{\Delta_\infty u}{|Du|^2}$. This creates several interesting PDE problems as listed at the end of the paper. Also, this might lead to a suitable interpretation of the parabolic infinity Laplacian equation.

In this paper, the authors also consider the case when the game is on a general metric space or a graph. Several interesting results are derived.

Yifeng Yu

From MathSciNet, April 2017

MR3304279 60K35; 60J67

Johansson Viklund, Fredrik; Sola, Alan; Turner, Amanda

Small-particle limits in a regularized Laplacian random growth model. (English summary)

Communications in Mathematical Physics **334** (2015), *no.* 1, 331–366.

It is a standard idea to describe growing clusters K_n using conformal maps Φ_n from the complement of the unit disc onto the complement of K_n. In this case it is natural to write $\Phi_n = f_1 \circ \cdots \circ f_n$ where f_k is a conformal map "adding" the kth particle to the cluster.

Hastings and Levitov proposed a family of models, called HL(α), that are defined using this framework. We assume that all functions $f_n(z)$ are of the form $e^{i\theta_n} f_{c_n}(e^{-i\theta_n} z)$, where θ_n are independent uniform rotations and f_{c_n} is the conformal map adding a straight slit of capacity c_n and length d_n (d_n and c_n are related by a simple algebraic formula) to the unit disc. For given α we choose $d_{n+1} = d/|\Phi_n'(e^{i\theta_{n+1}})|^{\alpha/2}$.

In the case $\alpha = 0$ we have that all d_n are the same and Φ_n is a composition of i.i.d. conformal maps. For $\alpha > 0$, this is no longer true since d_n depends on the entire past of the system. For $\alpha = 2$ the HL(2) model is related to the Diffusion Limited Aggregation (DLA) since the sizes of added "particles" are generally of the same order.

The usual HL models use the derivative of Φ_n on the boundary, which is usually ill-behaved. In the present paper the authors consider a regularized version, where the derivative is taken slightly away from the boundary. In terms of capacities they define

$$c_{n+1} = \frac{c}{|\Phi_n'(e^{\sigma + i\theta_{n+1}})|^\alpha},$$

where $\sigma > 0$ is a regularizing parameter.

It is believed that HL models have a phase transition at $\alpha = 1$: below it the clusters grow as a disc and for $\alpha > 1$ they have fractal structure.

In the present paper the authors study the small-particle limit ($c \to 0$) of the regularized HL(σ, α) model. The first result is that for any fixed $\alpha > 0$ and T, the functions Φ_N where $N = T/c$ converge uniformly on compacts to $(1 + \alpha T)^{1/\alpha} z$ as long as σ is much larger than $1/\sqrt{\log(1/c_n)}$. In particular it shows that the scaling limit is the disc for all values of α. This is interpreted as an indication that the regularization is too strong to preserve the fractal structure of the HL model for $\alpha > 1$.

In the second part of the paper, the authors describe the fine structure of the cluster in the double scaling limit $\alpha \to 0$ and $c \to 0$. Despite the fact that the scaling limit of Φ_n is deterministic and trivial, it turns out that it is possible to study the internal structure of the clusters. Following [J. R. Norris and A. G. Turner, Comm. Math. Phys. **316** (2012), no. 3, 809–841; MR2993934] the authors study the flow of harmonic measure, which can be interpreted in terms of the number of branches in the cluster. Under the assumption that the regularizing parameter does not go to zero too quickly, they show that when $\alpha \ll \sqrt{c}$ there is only one infinite branch, for $\alpha/\sqrt{c} \to a$ there is a random number of infinite branches and their number is stochastically increasing with a, and, finally, there is deterministic radial growth if $\alpha \gg \sqrt{c}$.

Dmitri B. Beliaev
From MathSciNet, April 2017

MR2400263 (2009b:60143) 60G50; 60K35

Levine, Lionel; Peres, Yuval

Spherical asymptotics for the rotor-router model in $\mathbb{Z}^d$. (English summary)

Indiana University Mathematics Journal **57** (2008), *no.* 1, 431–449.

In internal diffusion limited aggregation (IDLA) on $\mathbb{Z}^d$ a growing cluster is constructed as follows: particles are released at the origin one at a time, and each particle performs some motion on the integer lattice but freezes at the first unoccupied site it encounters. The questions of interest concern the asymptotic shape of the cluster of frozen particles. In this paper the individual particle motion is the deterministic rotor-router dynamics invented by J. Propp. Each site contains an arrow ("rotor") that gives the direction of the next step out of that site, and each visit of a walker updates the direction of the rotor by some deterministic rule.

The authors prove that under rotor-router dynamics the $n^{-1/d}$-scaled IDLA cluster converges to a Euclidean sphere. Convergence is weak in the sense that it is the measure of the symmetric difference that tends to zero. The assumption is that over the long term the update rule of the rotors cannot favor any particular lattice direction. The result is valid in all dimensions. For dimension $d = 1$ it was done earlier by one of the authors.

The proof makes clever use of random walk estimates. Roughly speaking, the deterministic rotor-router quantities can be related to random walk expectations because a large number of rotor-routings from a given site are almost equally divided between the $2d$ lattice directions. This enables a link with random walk expectations that themselves satisfy natural Laplacian identities.

The proof includes an isoperimetric inequality for simple random walk that shows, with quantitative bounds, that the lattice ball asymptotically maximizes expected exit time among sets of a given cardinality.

Timo Seppäläinen
From MathSciNet, April 2017

MR1203679 (94c:90132) 90D35; 05C20, 05C40

Björner, Anders; Lovász, László

Chip-firing games on directed graphs. (English summary)

Journal of Algebraic Combinatorics. An International Journal **1** (1992), *no.* 4, 305–328.

For the last six or seven years, a number of researchers have studied a certain graph-theoretical generalization of the West African games of Mancala. (See *Mancala games* by L. Russ [Reference Publ., Algonac, MI, 1984] for information on its 1200 known variants.) In these generalized "chip-firing games", a certain number of chips are placed on each node of a digraph. A node can be fired if the number of chips is at least equal to its out-degree. The effect of firing a node is to send one chip along each direction to its neighbors. By weighting edge, it is possible to consider more general mass-firing games.

Björner, Lovász, Shor, and Tardos had previously studied the case of undirected graphs. Given a graph with m edges and n nodes, the game is infinite with over $2m-n$ chips, and finite with under m chips. The finiteness of the intermediate cases depends not on the graph, but on the distribution of the chips within the graph. However, the situation for digraphs is not similar. If h is the number of disjoint directed cycles, then the existence of fewer than h chips leads to finite play. However, in general, the finiteness of play depends not only on the number of chips (and their placement), but also on the digraph itself.

Further results concern the reachability of a given position, and random walks in graphs. The shortest period of an infinite game is related to the length of the longest terminating game.

In his thesis, one of Björner's students, Kimmo Ericsson, generalizes these notions further, and introduces the notion of a number game (a problem inspired by the 1986 International Mathematics Olympiad). He suggests that two-player variants might be interesting to study, insofar as they do not possess the strong convergence property.

Daniel Elliott Loeb

From MathSciNet, April 2017

MR3012035 05-02; 05C60, 05C80, 05C82, 05D40

Lovász, László

Large networks and graph limits.

American Mathematical Society Colloquium Publications, Vol. 60.
American Mathematical Society, *Providence, RI*, 2012, xiv+475 pp.,
ISBN 978-0-8218-9085-1

This review, of an important new book by the renowned mathematician László Lovász, is not a review written by an expert or for experts, but rather one that seeks to interpret and facilitate understanding of some of the book's highlights for a general mathematical audience. The reviewer learned a lot as he studied the book. It is exquisitely written, and thus suitable for self-study, as well as for use in graduate seminars or courses. One of the features is that the reader can come away with a broad-strokes understanding of the material by reading the 30-page Introduction. No key points are skipped. Fittingly, the author often reminds readers of this fact by referring them back to the Introduction at appropriate points, usually just before the theory is rigorously developed.

The main ideas behind the book are a result of collaborations between the author and his colleagues at the Theory Group of Microsoft Research (headed by Jennifer Chayes) and elsewhere. The algebra of graph homomorphisms, and the analytic theory of convergence of graph sequences, both dense and of bounded degree, form the heart of the book. There are connections that appear continually, to branches of mathematics such as measure theory and functional analysis ("measurability carries combinatorial meaning"), probability and statistics (property testing, or statistics on graphs), extremal graph theory (particularly Turán theory and the Szemerédi regularity lemma), representation theory of algebras, and so forth. The limit objects for convergent graph sequences are called *graphons* and *graphings* in the *dense* and *bounded degree* cases respectively. The limiting objects may be thought of as graphs on the continuum rather than graphs on a countable set, and two baseline reasons for this fact are that (i) one cannot construct, in a nontrivial way, an uncountably infinite family of independent random variables, and (ii) there is no way to define a uniform distribution on a countable set.

Large graphs and large networks appear everywhere, as hyperlink graphs on the internet, social network graphs, crystal graphs in statistical physics, chip design graphs, Erdős-Rényi random graphs, quasirandom graphs, randomly growing graphs, and so forth. These examples encompass dense graphs with large vertex degrees that are a positive fraction of the nodes; bounded degree graphs; and graphs in which the maximum degree tends to be large but with a vanishing fraction of the edges being present. Only the first two cases are systematically studied in the book.

Obtaining information about large graphs may be done through sampling small subgraphs, which is done quite easily in the dense case through possibly repeated *subgraph sampling*. Instead of considering randomly selected induced subgraphs, one might, almost equivalently, define a *homomorphism density*, which is the probability that a random map from the vertex set $V(F)$ of one graph to the vertex set $V(G)$ of another preserves adjacency. The author develops both of these methods of obtaining information, but the book focuses more on homomorphisms, which allow one, through left and right homomorphisms from a small graph F, and to a small graph H as illustrated through the scheme

$$F \longrightarrow G \longrightarrow H,$$

to study both sampling and global observables. The distance between two large graphs may be measured in several ways; the author introduces sampling distance and cut distance, and shows how a complex graph's pixel picture may be suitably rearranged to reveal the kind of structure guaranteed by the Szemerédi regularity lemma.

Lovász next introduces graph convergence and limits, for both dense graphs and those with bounded degree. Suppose we sample graphs of fixed size k from G_n, and consider the limiting distribution as $n \to \infty$. If the limiting distribution exists for each k, the sequence G_n is called *locally convergent*. It turns out that either the cut distance or the sampling distance may be used to exhibit convergence. But what does the sequence converge to? In the dense case, the limit objects are identified in the Introduction as measurable two variable functions $W\,[0,1]^2 \to [0,1]$ that are called graphons—for example random graphs $G(n, 1/2)$ converge to the constant function 1/2. Though graphons can be very complicated, in many cases the limiting density of a growing graph sequence is a graphon that is continuous and has a simple formula.

Sampling provides a means of estimating parameters such as the proportion of triples that form a triangle. This is one example of how one may run algorithms on large graphs. Others include property testing, e.g., whether the graph is planar; the computation of a structure whose size might be comparable to that of the graph; and selecting a representative set of nodes that each node in the graph is "similar" to.

In the bounded degree (BD) case, a sampled graph is most likely to be edgeless. Neighborhood sampling may be used to rectify this, but there are other possibilities. For BD graphs, sampling distance is defined analogously, but with neighborhood distributions being used in place of sampling distributions. An important notion in this theory is that of r-neighborhood densities, which are obtained by conducting a depth r exploration of the neighborhood of a randomly selected node. The notion of convergence in the dense case is actually modeled on notions of convergence for BD graphs introduced early by I. Benjamini and O. Schramm [Electron. J. Probab. **6** (2001), no. 23, 13 pp.; MR1873300]. However, the author introduces a stronger notion of convergence in the BD case, called *local-global*, and to objects called graphings, first introduced by group theorists. Finally, the author argues in the first chapter of his Introduction that the algorithmic theory of computing a structure is as advanced as in the dense case.

Homomorphism densities are fundamental to the development in the book, so in the second introductory chapter the definitions of Chapter 1 are reinforced for large graphs in terms of classical Turán theory and hypergraph theory. An algebraic proof of Goodman's result from 1959 regarding triangle densities is provided using "little pictograms", and advance notice is given that this proof is quite rigorous. The chapter ends with a discussion of Ising models from statistical physics.

The first two chapters of Part 2 of the book introduce the basic terminology and notation for the algebra of graph homomorphisms and provide definitions of graph classes such as weighted and signed graphs, operations on graphs, and types of graph properties. Connection matrices are introduced and many examples of graph parameters with finite connection rank are given. Two key theorems that guarantee finite connection rank are proved.

The author next moves on to the heart of the theory of the algebra of graph homomorphisms. Some of the key issues addressed are as follows:

(i) When do graph homomorphisms exist between two graphs (written $F \to G$)? For example, if $K_n \to G$, then G has an n-clique.
(ii) What are homomorphism numbers and why are they differently defined for non-simple graphs? (In the simple graph case $\hom(F,G)$ is the number of homomorphisms between F and G.)
(iii) What are regular, injective and induced homomorphism densities $t(F,G)$, $t_{\mathrm{inj}}(F,G)$, and $t_{\mathrm{ind}}(F,G)$ respectively? (In the dense case we have the formula $t(F,G) = \hom(F,G)/n^k$, where n, k are the number of vertices in G, F respectively.)
(iv) Why does this definition have to be altered in the BD case?
(v) What is the connection between homomorphism densities and sampling?

Examples of many parameters that can be expressed in terms of homomorphism numbers are given. The fact that $\hom(\cdot, G)$ determines G is recalled, the algebraic properties of graph multiplication are given, and the independence of homomorphism functions is addressed. Theorem 5.54 provides a necessary and sufficient condition for a graph parameter to equal $\hom(\cdot, G)$ for some weighted graph G. Homomorphism numbers into randomly weighted graphs are defined, allowing for an extension of Theorem 5.54. Chapter 5 ends with a brief discussion of the structure of the homomorphism set.

Quantum graphs are formal linear combinations of multigraphs with real coefficients. Graph parameters can be extended to quantum graphs, which allow one to conveniently express, e.g., combinatorial situations such as the contraction deletion relation of the chromatic polynomial. Algebras on the infinite-dimensional vector space of labeled quantum graphs include the gluing and the concatenation algebras, whose structure is explored. These developments allow for a complete proof of Theorem 5.54. *Contractors* and *connectors* are defined as proper expansions of partially labeled graphs modulo a graph parameter, and a characterization is given for those parameters that have contractors or connectors; these parameters include the homomorphism functions. The rest of the chapter studies algebras for homomorphism functions, and, as an application, it is proved that graph parameters with finite connection rank can be computed in polynomial time.

Part 3 of the book is about dense graph sequences and provides the well-developed theory of graph limits in the dense case. In Chapter 7, graph functions, or graphons, are defined as bounded symmetric measurable functions from $[0,1]^2$ to $[0,1]$; graphons with values in $\{0,1\}$ can be considered to be graphs with vertex set $[0,1]$. *Kernels* are symmetric functions with range space equal to $\mathbb{R}$. The subfamily of stepfunctions is defined in a natural way, and corresponds to weighted graphs on finite sets; the stepfunction W_H corresponding to a weighted graph H depends on the way H is labeled. Homomorphism densities in graphs extend to homomorphism densities in graphons, or, in fact, to kernels. The first step is to define $t(F, W)$, where F is a multigraph and W a kernel, and then, for a weighted graph G we set $t(F, G) = t(F, W_G)$. It is proved that the graph parameter $t(\cdot, W)$ is multiplicative and reflection positive for each graphon W. Weak isomorphism of two kernels is defined in terms of measure preserving maps and the distinction between weak and regular isomorphism is stressed. The next section of the chapter shows how each kernel can be written as the direct sum of "connected kernels", which provides us with an analog of graph decomposition. Likewise, the pointwise, operator and tensor products of kernels are defined. Simple graph properties such as bipartiteness and triangle-freeness are shown to generalize to graphons in a simple fashion. The chapter ends with a discussion of kernel operators $T_W\ L_1[0,1] \to L_\infty[0,1]$, which, if we (meaningfully) consider $T_W\ L_2[0,1] \to L_2[0,1]$, are Hilbert-Schmidt.

Chapter 8 begins with a definition of the cut norm of a matrix, and goes on to define the edit (ℓ_1) and cut distances between two graphs in terms of their adjacency matrices. This leads to the notion of cut distance between two arbitrary graphs on different numbers of nodes. The cut norm and cut distance of kernels are then defined, and key properties are uncovered. Finally, the relationship between the cut and L_1 norms is explored.

The next chapter reviews the Szemerédi regularity lemma for large dense graphs, in its original, "weak", and "strong" forms. The weak version of the lemma is then formulated for kernels, and stated in terms of the cut distance to stepfunctions.

The strong regularity lemma is then proved for graphons. Both of these are then seen to be a consequence of Theorem 9.23, which states that (identifying graphons at cut distance zero) the set of graphons is compact in the cut distance. These extensions are strikingly simple and elegant.

Chapter 10 deals with the analysis of sampling from a graph. In the first section, for any graphon W, W-random graphs are introduced as a generalization of Erdős-Rényi random graphs, and lead to a description on what it means to sample from a graphon. W-random graphs are then used to define the sampling distance between graphons, and an upper bound is derived for the difference between the sampling distance of graphs, and that of the sampling distance between the associated graphons. An Azuma-type inequality is then proved for the concentration of "reasonably smooth" parameters of a random induced subgraph obtained by sampling, and an analog is given for graphons. The key sampling result, Lemma 10.5, exhibits the fact that the sampling distance between two graphs can be well estimated by sampling. A second such result estimates the cut distance between a graph and a random induced graph of size k obtained by sampling. These results are then extended to comparisons between a graphon and its associated W-random graph. The chapter finishes with "counting lemmas" that relate homomorphism densities (equivalent to sampling distributions) to cut distance, both directly and in a converse form.

Chapter 11 addresses the central theme of the book, namely convergent graph sequences and their limits. In the dense case, we use either subgraph sampling or homomorphism densities to define "left-convergence". A sequence $\{G_n\}$ of simple graphs whose node sequence tends to infinity is said to converge if for any graph F, $\{t_{F,G_n}\}$ is a Cauchy sequence. The counting and inverse counting lemmas quickly reveal that a sequence $\{G_n\}$ of simple graphs whose node sequence tends to infinity is convergent iff it is Cauchy in the cut distance. The first *construction* of the limiting object is called the weak limit. Here we look at the distribution $\sigma_{G,k}$ of the random graph on k vertices obtained by sampling. Now if the sequence G_n is convergent then the distributions $\sigma_{G_n,k}$ tend to a limit σ_k for each k (and conversely), and thus the sequence $\{\sigma_k : k \geq 1\}$ encodes the "limit" of the convergent graph sequence. Moreover, this sequence forms a consistent set of measures (in the sense of Kolmogorov consistency) called a *local random graph model*. In addition, it is shown that all these measures σ_k can be combined into a single distribution called a *countable random graph model*. An explicit description of the limit object is given in Theorem 11.21: For each convergent simple graph sequence G_n, there exists a graphon W such that $t(F, G_n) \to t(F, W)$ for each simple graph F; W is called the limit of G_n. A large number of examples are provided, e.g. simple threshold graphs, quasirandom graphs, preferential attachment graphs, etc. The important Theorem 11.52 proves that the various formulations of graph limits, including graphons, local and consistent random graph models, and local countable random graph models, are all "cryptomorphically" equivalent. Convergence of the spectra of the convergent sequence G_n to that of W is addressed in Section 11.6.

The next chapter completes the agenda of considering large graphs from "both sides", by studying homomorphisms from large graphs into small ones, which naturally arose before in several contexts, e.g., the number of k-colorings of graphs, and Ising models in statistical physics. Does, e.g., the fact that $\hom(\cdot, G)$ and $\hom(G, \cdot)$ have equivalent characterizations lead to a theory of right graph convergence? A naïve theory leads to notions that help, in Theorem 12.12, to formulate a characterization of convergence of graphon sequences in terms of these notions (e.g., overlay functional values). An analog is given in Theorem 12.20 for convergent dense graph sequences. The material in this chapter is based on very recent work of C. Borgs et al. [Ann. of Math. (2) **176** (2012), no. 1, 151–219; MR2925382].

Chapter 13 deals with the general form of kernels and graphons viewed as symmetric measurable functions on $J \times J$, where $J = (\Omega, \mathcal{A}, \pi)$ is a probability space. Subgraph densities and weak isomorphism are defined as before, but the generalization of cut distance needs more care. It is shown that a kernel on an abstract probability space can be transformed to one on a standard probability space. It turns out that choosing the underlying probability space judiciously leads to simpler and more convenient forms for W; examples are given to illustrate this fact. The rest of the chapter (i) builds a theory of pure kernels, twin-free kernels, and density functions on pure kernels, (ii) studies the topology of a graphon (e.g., for pure kernels, the so-called similarity distance defines the weak topology on the completion of J in the weak topology), and (iii) examines symmetries of graphons.

The structure of the space of graphons is studied in Chapter 14. This space is where the interplay between graph theory and analysis is most evident. Norms can be defined in terms of homomorphism densities, and we may ask for which graphs F, e.g., the functional $W \mapsto |t(F, W)|^{1/e(F)}$ is a norm or seminorm on the space of kernels; such Fs are called *norming* or *seminorming*. Various facts concerning graphs that are (semi-)norming are studied. Bipartite graphs $K_{n/2,n/2}$ are norming, and a graph is seminorming iff it has the *Hölder property*. In Section 14.2, the author studies the topologies on the graphon space defined by the cut and L_1 norms, and some relationships between them. In Section 14.3, closures of graph properties such as those that are hereditary or random-free are studied. For example, we have that the closure of a hereditary graph property $\mathcal{P}$ is characterized by the equations $t_{\text{ind}}(F, W) = 0$ for each $F \notin \mathcal{P}$. The author next moves on to *graphon varieties*, i.e. subsets of graphons specifying linear dependence between subgraph densities. For example, the equation $t(K_3 - 2K_2K_2 + K_2, W) = 0$ yields the variety consisting of the graphons W_{K_n} and the constant graphon. A theory of graphon varieties is then built. The last section is devoted to a study of random graphons. We define probability measures on the sigma algebra of Borel subsets of equivalence classes of graphons to be a *random graphon model*. Other equivalent definitions exist, as seen in Theorem 14.59.

There are many algorithmic questions that may be asked for large graphs. In Chapter 15, Lovász begins with parameter estimation, characterizing those parameters that are estimable in a precise sense related to sampling. An example of such a parameter is the normalized maximum cut. Property testing is the next key notion studied, with Theorem 15.8 providing a characterization of graph properties that are distinguishable by sampling. Arbitrary probability numbers of 2/3 and 1/3 are used to "distinguish" whether a graph has property $\mathcal{P}_1$ or $\mathcal{P}_2$, based on a third property $\mathcal{Q}$. Testing for a single property is a much harder problem, as described in Section 15.3. Attention is focused on those aspects that are related to graph limit theory. Testable graphon properties are defined, and a series of results presented that characterize testability; for example, Theorem 15.16 states that a closed graphon property $\mathcal{P}$ is testable iff the functional $d_1(\cdot, \mathcal{P})$ is continuous in the cut norm. Testable graph properties are then shown to be those that are *robust* and whose closures are testable; conversely, a graphon property is testable iff it is the closure of a testable graph property. Connections are given to hereditary graph properties. Computable structures are studied in Section 15.4, with special attention given to algorithms for computing weak Szemerédi partitions in huge graphs, and to approximately compute the max cut.

Extremal graph theory for dense graphs is an excellent area of applications of graph limits, and Chapter 16 deals with this issue. The technical tools of reflection positivity and variational calculus are developed in the first two sections. There has been extensive research on the relationships of complete graph densities in a graph G, and Section 16.3 contains a survey of results in the context of graph limits. A key result in this area is that for a quantum graph g whose constituents are complete graphs, we have that $t(g, W) \geq 0$ for each graphon (written as $g \geq 0$) iff $t(g, K_n) \geq 0$, $n \geq 1$. Many corollaries emerge, including a Turán theorem for graphons. The exact relationship between edge and triangle densities is described next. In particular, Razborov's theorem, originally proved using Razborov's flag algebra technique, is proved. The theorem can be stated directly as follows: If G is a graph with $t(K_2, G) = d$, then $t(K_3, G) \geq f(d)$ for a certain f. Next, a general theorem that generalizes Turán's theorem is stated and proved using graph limits. In Section 16.5, a result of N. Alon and U. Stav [Random Structures Algorithms **33** (2008), no. 1, 87–104; MR2428979] is generalized to graphons: the farthest d_1 distance from a flexible graphon property is attained by a constant function. Sidorenko's conjecture [A. F. Sidorenko, Diskret. Mat. **3** (1991), no. 3, 50–65; MR1138091; Graphs Combin. **9** (1993), no. 2, 201–204; MR1225933], formulated in terms of integral representations (i.e. graphons), is still unproven in general, but significant progress is made by characterizing graphs that satisfy the "local Sidorenko property". The exposition continues with decidability of density inequalities: H. Hatami and S. Norine [J. Amer. Math. Soc. **24** (2011), no. 2, 547–565; MR2748400] proved that no certificate of non-negativity can be found for quantum graphs. However, Theorem 16.41 gives a necessary and sufficient condition for non-negativity in terms of square-sum g's, the so-called Positivstellensatz for graphs. The chapter ends with a discussion of which graphs are extremal. Since all classical extremal problems have a solution whose "template" is a stepfunction, since stepfunctions are known to be "finitely forcible", and since the finitely forcible graphon is the unique solution to a general graph theoretic extremal problem, it follows that each stepfunction is the template of an appropriate extremal problem. Theorem 16.46 and Corollary 16.48 give specific details.

This part on dense graph limits ends in Chapter 17 with limit objects being defined for multigraphs, directed graphs, colored graphs, hypergraphs, etc. We skip the details.

In the next part of the book, regarding bounded degree (BD) graphs, all degrees are assumed to be bounded by D. Graphons, symmetric functions of two variables, have been studied for many reasons for many years. Graphings, on the other hand, have been studied just occasionally before, by group theorists. Let $\mathbf{G}$ be a graph with node set $V(\mathbf{G}) = \Omega$, where Ω is uncountable. $(\Omega, \mathcal{B})$ is a Borel sigma algebra. $\mathbf{G}$ is called a Borel graph if its edge set is a Borel set in $\mathcal{B} \times \mathcal{B}$. We know that each Borel graph has a Borel coloring with $D + 1$ colors (an analog of Brooks' theorem) and a Borel edge coloring with $2D - 1$ colors (an analog of Vizing's theorem). These notions tie in with the definition of a graphing: A graph $\mathbf{G}$ with node set $V(\mathbf{G}) = \Omega$ is *measure preserving,* or a *graphing* if it is Borel, and $\int_A \deg_B(x) d\lambda(x) = \int_B \deg_A(x) d\lambda(x)$. As an example, let F be a finite graph and define $\mathbf{G}_F$ as follows: Let $V(F) = [n]$ and split $[0, 1)$ into the n intervals $J_i = [(i-1)/n, i/n)$. If $ij \in E(F)$, $i < j$, connect each $x \in J_i$ to $x + (j-i)/n \in J_j$. It may be verified that $\mathbf{G}_F$ is a graphing, and it is noteworthy that its picture is similar to the pixel picture of the associated graphon, except that black squares are replaced by white squares with a diagonal. If $\mathbf{G} = (\Omega, \mathcal{B}, \lambda, E)$ is a graphing, then so is $bfG' = (\Omega, \mathcal{B}, \lambda, L)$, where $L \subseteq E$ is a symmetric Borel set. This fact provides, in Theorem 18.21, a close connection between graphings and measure preserving families supported by graphs.

After the above introductory Chapter 18, the author moves on to the theory of convergence for BD graphs. Now there is no good analog of cut distance, so the weaker notion of sampling distance between two graphs is used instead. First, *sampling distance at depth* r is introduced, and then the weighted sum of these, over r, yields the sampling distance. Various inequalities relating this distance to the edit and variation distance are presented. There are two non-equivalent and quite reasonable definitions of convergence, and the author treats both. *Local convergence* is introduced first: A sequence G_n of graphs with vertex sizes tending to infinity is said to be locally convergent if the r-neighborhood densities introduced earlier, obtained from the root node by conducting a depth r exploration of the graph, and denoted by $\rho_{G_n}(F)$, converge for each r and for each r-ball F. It turns out that every locally convergent graph sequence gives rise to an involution invariant distribution on the sigma algebra on rooted graphs. This is the local limit, introduced by Benjamini and Schramm. Thus there is a graphing $\mathbf{G}$ such that $\rho_{G_n,r} \to \rho_{\mathbf{G},r}$ for each r, written as $G_n \to \mathbf{G}$, and we say that $\mathbf{G}$ represents the limit. For example, if G_n is the $n \times n$ grid, then the Benjamini-Schramm limit is concentrated on a rooted infinite square grid, which the author shows can be represented as a graphing. Interestingly, Conjecture 19.8 reveals that we don't know whether each involution invariant distribution on rooted graphs is the limit of a locally convergent BD graph sequence. Finally, this chapter defines the second notion of convergence, namely local-global (LG) convergence. This idea depends on the notion of *non-deterministic sampling distance of depth* r *for* k*-colored graphs*, and the local-global limit is the graphing $\mathbf{G}$ if this distance between G_n and $\mathbf{G}$ tends to zero. Theorem 19.16 reveals that for every LG convergent sequence of finite graphs, there is a graphing $\mathbf{G}$ that is its LG limit.

The rest of this part on BD graphs deals with right convergence, the structure of graphings, and algorithms for BD graphs. Some of the highlights are as follows:

(i) An account of the theory of random homomorphisms, from a countable graph or graphing into a weighted graph, developed by statistical physicists. The exposition includes examples such as the Ising model, defines random homomorphisms from a BD countable graph into a graphon (and the corresponding *homomorphism entropy*), and builds, in Theorem 20.15, a key link between local convergence of G_n and the convergence of the homomorphism entropy $\mathrm{ent}(G_n, W)$ for a class of graphons W.

(ii) A discussion, with examples, of *hyperfinite graph families*, ones in which deletion of a certain number of edges leads to components of bounded size: Given are a brief proof of the fact that practically all minor closed families are hyperfinite; a development of the theory of hyperfinite graphings; and Theorem 21.13, which proves the equivalence of hyperfiniteness of a convergent graph sequence and that of its limit graphing.

(iii) A summary of the extensive algorithmic theory for large BD graphs. Topics covered include: a characterization of estimability for BD graph parameters in terms of locally convergent graph sequences; estimability of the number of spanning trees; a criterion for the distinguishability by sampling of two graph properties; the role of hyperfiniteness in testability; and an extensive discussion of computable structures in BD graphs.

Extensions to sparse but not too sparse graphs, edge coloring models, hypergraphs, and graph categories form the content of the last part.

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Sandpiles, spanning trees, and plane duality. (English summary)

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This work seems to have originated in a consideration of sand migration between dunes but evolved considerably; I shall describe four stages.

Stage 1. Consider a connected multigraph G without loop edges; imagine a chip-exchanging game with an initial state in which, at each vertex v, there are a_v chips where $a_v \in \mathbb{Z}$; thus the state is modeled as an element of the abelian group $\mathrm{Div}(G) = \mathbb{Z}^{|V(G)|}$; the subgroup in which the total number of chips is zero is written $\mathrm{Div}^0(G)$. Now suppose that a move consists in a particular player, v, distributing one chip from his 'pile' along each edge incident with v. This move can be considered as an element D of $\mathrm{Div}^0(G)$; the subgroup generated by such elements is denoted by $\sim$, and the *sandpile group* of G is $\mathcal{S}(G) = \mathrm{Div}^0(G)/\sim$. The coset corresponding to D is written $[D]$. The authors note that $|\mathcal{S}(G)|$ is equal to the number of spanning trees of G [R. Bacher, P. de la Harpe and T. Smirnova-Nagnibeda, Bull. Soc. Math. France **125** (1997), no. 2, 167–198; MR1478029].

Stage 2. An orientation is fixed on each edge of G; also, the edges incident with any vertex v are given a fixed cyclic ordering. Now consider the abelian group $\mathbb{Z}E = \mathbb{Z}^{|E(G)|}$; then any directed cycle on G corresponds to some element of $\mathbb{Z}E$, and the group generated by these is the *cycle space*, $\mathcal{C}$. Next, for any subset $U \subset V$, the set of all edges that join a vertex of U to one of $V \setminus U$ is a *cut*; by directing these edges away from vertices in U and comparing the result with the fixed orientation, we obtain an element of $\mathbb{Z}E$, and the *cut space* $\mathcal{C}^*$ is the subgroup of $\mathbb{Z}E$ generated by these cuts. Define $\mathcal{E}(G)$ to be the quotient group $\mathbb{Z}E/(\mathcal{C} + \mathcal{C}^*)$, and define the *boundary map* $\mathbb{Z}E \to \mathrm{Div}^0(G)$ by mapping each edge $e = (u, v)$ to the element $v - u$. The authors show that this defines an isomorphism $\partial_G \colon \mathcal{E}(G) \to \mathcal{S}(G)$.

Stage 3. We play a new game in which at each move some vertex v is called, which 'fires' a chip along *one* incident edge; it then remembers which edge has been used, and uses the cyclically next edge the next time it is called. There is one vertex, q, which is never asked to fire. The function $\rho \colon V \setminus \{q\} \to E$ describing last-used edges is the *rotor configuration.* Previous work has shown that for any initial $D \in \mathrm{Div}^0(G)$ and rotor configuration ρ there is a firing sequence that results in the state 0, the final rotor configuration being a spanning tree T directed into q, depending on $[D]$ and written $[D] \cdot T$. For $\mathcal{T}(G)$ denoting the set of spanning trees of G, the mapping

$$\mu_G\colon ([D],T) \mapsto [D]\cdot T \quad ([D]\in \mathcal{S}(G), T\in \mathcal{T}(G))$$

is a simply transitive action of $\mathcal{S}(G)$ on $\mathcal{T}(G)$.

Stage 4. Now suppose that G is bridgeless and that the cyclic orderings of the edges at the vertices are such that $G=(V,E)$ is planar; let $G^*=(V^*,E^*)$ be the planar dual. Since the cycles of G are cuts of G^* and vice versa, we obtain an isomorphism $\epsilon\colon \mathcal{E}(G)\to\mathcal{E}(G^*)$ and an isomorphism $\phi\colon \mathcal{S}(G)\to\mathcal{S}(G^*)$ such that $\partial_{G^*}\circ\epsilon=\phi\circ\partial_G$ (i.e., we have a commutative diagram).

The main theorem can now be described in terms of another commutative diagram. There is a bijection $\delta\colon \mathcal{T}(G)\to\mathcal{T}(G^*)$ between the spanning trees of G and G^*, and the theorem states that $\delta\circ\mu_G=\mu_{G^*}\circ(\phi\times\delta)$. In words, the rotor-rotating action is compatible with plane duality.

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