BOOK REVIEWS

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Yosida approximations of stochastic differential equations in infinite dimensions and applications, by T. E. Govindan, Probability Theory and Stochastic Modelling Series, Vol. 79, Springer, Cham, 2016, xix + 407 pp., ISBN 978-3-319-45682-9

1. Introduction

Stochastic partial differential equations are simply partial differential equations in the presence of uncertainty. Uncertainty, in its simplest form, is modeled by (or taken as) the time derivative (in the sense of distributions) of a Wiener process, known commonly as white noise. The introduction of a random force in a partial differential equation (PDE) arises from the need to explain the fluctuations observed in physical phenomena and to account for external disturbances and measurement errors.

Randomness is also introduced in order to take advantage of certain special methods and tools in stochastic analysis, such as the Stroock–Varadhan martingale problems and the Girsanov transformation, that do not have a counterpart in the theory of partial differential equations. This opens up the possibility of solving partial differential equations when perturbed by a noise term that are otherwise unsolvable. The addition of a noise term also allows one to study problems, such as the existence of an invariant measure, ergodic behavior of solutions, and the large deviation principle, which do not have meaning in a deterministic setting.

A stochastic partial differential equation, in an abstract evolution setup, is an infinite-dimensional stochastic differential equation. To explain the various terminologies and stochastic analysis of such equations, we start with a complete probability space $(\Omega, \mathcal{F}, P)$, where Ω is an abstract space, $\mathcal{F}$ is a σ-field of subsets of Ω, and P is a probability measure on $\mathcal{F}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be an increasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$, and let $\mathcal{F}_t = \bigcap_{r>t} \mathcal{F}_r$ for all $t \geq 0$. We will call $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ a stochastic basis.

Let X and Y be two real separable Hilbert spaces. Consider the following basic X-valued stochastic differential equation on $[0, T]$:

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dW(t), \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

Here, W is a Y-valued Wiener process with a nuclear covariance operator Q. The coefficient f maps X to X, and $g : X \to L(Y, X)$, where $L(Y, X)$ denotes the

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space of all bounded linear operators from Y to X. The initial condition x_0 is an X-valued, $\mathcal{F}_0$-measurable random variable.

Equation (1.1) is understood as an integral equation

$$x(t) = x_0 + \int_0^t f(x(s))ds + \int_0^t g(x(s))dW(s) \text{ a.s.,} \tag{1.2}$$

with the stochastic integral taken in the sense of Itô. A stochastic process x is called an $\{\mathcal{F}_t\}$-adapted process, or simply $\mathcal{F}_t$-adapted, if $x(t)$ is an $\mathcal{F}_t$-measurable random variable for each $t \geq 0$. If an $\mathcal{F}_t$-adapted process x satisfies certain integrability requirements and solves (1.2) for all $t \in [0, T]$, then x is called a *strong* solution of (1.1). A strong solution is called pathwise unique if any two solutions x and y satisfy

$$P\{x(t) = y(t) \ \ \forall t \in [0, T]\} = 1.$$

If the given stochastic basis and the Wiener process are changed in order to establish the existence of a solution, then such a solution is known as a *weak* solution. The concept of a weak solution in stochastic analysis should not be confused with that in PDE theory. The need for changing the given stochastic basis arises primarily due to

(i) removal of drift by a change of measures, or
(ii) application of the Skorohod representation theorem that converts convergence in law to almost sure convergence.

The Itô formula for the strong solution x is given by

$$h(x(t)) = h(x_0) + \int_0^t Lh(x(s))ds + \int_0^t \langle h_x(x(s)), g(x(s))dW(s)\rangle, \tag{1.3}$$

where $h : X \to R$ is twice continuously Fréchet differentiable in x and the operator Lh is defined by

$$Lh := \langle h_x, f\rangle + \frac{1}{2}\mathrm{tr}\,[h_{xx} g Q g^*].$$

In particular, if $h(x) = \|x\|^2$, (1.3) gives the energy equality which plays a central role in the solvability of (1.1).

2. Stochastic evolutions and stability theory

An X-valued stochastic evolution equation appears in the form

$$\begin{aligned} dx(t) &= [Ax(t) + f(x(t))]dt + g(x(t))dW(t), \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}$ on X. Let us assume that the coefficients satisfy a global Lipschitz condition of the form

$$\begin{aligned} \|f(x_1) - f(x_2)\| &\leq c_1\|x_1 - x_2\|, \\ \|g(x_1) - g(x_2)\|_{L_2^0} &\leq c_2\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X, \end{aligned}$$

where $\|\cdot\|$ denotes the norm on X, and $\|\varphi\|_{L_2^0}^2 := \mathrm{tr}\,(\varphi Q \varphi^*)$. The initial condition x_0 is taken as an X-valued, $\mathcal{F}_0$-measurable random variable with $E[\|x_0\|^p] < \infty$ for some $p \geq 2$.

A *mild solution* of (2.1) is defined as an adapted process x such that $\int_0^T \|x(t)\|^2 dt < \infty$, and for each $t \in [0, T]$,

$$x(t) = S(t)x_0 + \int_0^t S(t-r)f(x(r))dr + \int_0^t S(t-r)g(x(r))dW(r) \text{ a.s.} \tag{2.2}$$

The concept of mild solutions, in contrast to strong and weak solutions, avoids the unrealistic requirement that $x(t)$ takes values in the domain of A for all t. A strong solution of (2.1) is also a mild solution, and the converse is, in general, not true.

Under the stated conditions, a unique mild solution exists. However, the Itô formula is not available for it since (2.2) is an equation of the Volterra type (in infinite dimensions!). To overcome this hurdle, the author, inspired by Ichikawa [4], introduces the approximation

$$\begin{aligned} dx(t,\lambda) &= \{Ax(t,\lambda) + R(\lambda)f(x(t,\lambda))\}dt + R(\lambda)g(x(t,\lambda))dW(t), \\ x(0,\lambda) &= R(\lambda)x_0, \end{aligned} \tag{2.3}$$

where $R(\lambda) = \lambda R(\lambda, A)$, and $R(\lambda, A)$ is the resolvent operator $(\lambda I - A)^{-1}$ for λ in the resolvent set $\rho(A)$. Equation (2.3) admits a strong solution and serves as an *alternative* to the equation

$$\begin{aligned} dx_\lambda(t) &= \{A_\lambda x_\lambda(t) + f(x_\lambda(t))\}dt + g(x_\lambda(t))dW(t), \\ x_\lambda(0)) &= x_0, \end{aligned}$$

where A_λ is the Yosida approximation of A, given by $A_\lambda = AR(\lambda)$. Both $x(\cdot, \lambda)$ and x_λ converge to the mild solution x in $C([0,T]; L^p(\Omega, \mathcal{F}, P; X))$ as $\lambda \to \infty$. The author calls $x(\cdot, \lambda)$ the Yosida approximation for x. The infinitesimal generator for the $x(\cdot, \lambda)$ process is given by

$$L_\lambda h = \langle h_x, A + R(\lambda)f\rangle + \frac{1}{2}\mathrm{tr}\{h_{xx}(R(\lambda)g)Q(R(\lambda)g)^*\}.$$

Yosida approximations facilitate the study of quantities such as $\mathbb{E}h(x(t))$ by first calculating $\mathbb{E}h(x(t,\lambda))$ and then by letting $\lambda \to \infty$. Indeed, by the Itô formula,

$$\mathbb{E}h(x(t,\lambda)) = \mathbb{E}h(R(\lambda)x_0) + \mathbb{E}\int_0^t L_\lambda h(x(s,\lambda))ds.$$

A term-by-term passage of limits in the above equation is not possible because of the appearance of $\lim_{\lambda\to\infty}\langle h_x, Ax(t,\lambda)\rangle$. This limit cannot be written as $\langle h_x, Ax(t)\rangle$ since there is no guarantee that $x(t)$ would take values in the domain of A. Such hurdles are avoided in the study of stability of solutions by finding suitable functions h for which $L_\lambda h$ is bounded above by a linear function of h. A simple result of this type is as follows.

If $h : X \to \mathbb{R}$, such that h and its Fréchet derivatives h_x and h_{xx} are continuous and satisfy

(i) $|h(x)| + \|x\|\,\|h_x(x)\| + \|x\|^2\,\|h_{xx}(x)\| \le c\|x\|^p$ for some $p \ge 2$, $c > 0$, and
(ii) $L_\lambda h(x) + \alpha h(x) \le 0$ for an $\alpha > 0$ and for all $x \in D(A)$,

then

$$\mathbb{E}h(x(t)) \le \mathbb{E}[h(x_0)]e^{-\alpha t}. \tag{2.4}$$

If $\beta h(x) \ge \|x\|^p$ for some $\beta > 0$ and $p \ge 2$, then

$$\mathbb{E}\|x(t)\|^p \le \beta\mathbb{E}[h(x_0)]e^{-\alpha t}. \tag{2.5}$$

Such a result is called *exponential stability* of the pth moment.

In several problems, the decay of solutions may be at a slower than exponential rate, which leads one to study $\limsup_{t\to\infty} \frac{\log \mathbb{E}\|x(t)\|^p}{\log \lambda(t)}$ for a suitable decay function λ. As above, if a function $h(t,x)$ exists that satisfies a set of conditions, such as $(Lh)(t,x) \le c_1 + c_2 h(t,x)$ for two constants c_1 and c_2, and if $\|x\|^p \lambda^m(t) \le h(t,x)$ for an $m > 0$, then one is able to show that

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}\|x(t)\|^p}{\log \lambda(t)} \le -m.$$

That is, the pth moment of the solution decays at the rate $\lambda(t)^{-m}$.

Exponential stability of almost every sample path of $h(x(t))$ and the almost sure behavior of $\limsup_{t\to\infty} \frac{\log\|x(t)\|}{\log \lambda(t)}$ can also be studied by using the Itô formula and a Borel–Cantelli argument.

A different type of stability of a solution x is known as stability in distribution. As the solution x is a Markov process, it makes sense to look for the existence of a probability measure π on X such that the transition probability measure $p(t, x_0, dz)$ converges weakly to π as $t \to \infty$ for any initial value $x_0 \in X$. A natural consequence of this is the existence of a unique invariant measure for x, which implies the ergodicity of x (cf. [3]).

3. The content and format of the monograph

The book under review presents a systematic study of the convergence of Yosida approximations for infinite-dimensional stochastic evolution equations to the mild solution of the equation. The author establishes the convergence for a variety of stochastic evolutions driven by a Wiener process and/or a compensated Poisson random measure. This includes stochastic equations with delay, McKean–Vlasov equations, multivalued and controlled stochastic partial differential equations, and equations with Markovian switching.

Following upon the pioneering work of Ichikawa [4], the book presents several results on stability theory as a consequence of Yosida approximations. Exponential stability and stabilizability, robustness in stability for stochastic delay equations, and stability in distribution are shown for the various types of equations mentioned above.

Next, stochastic optimal control problems are discussed as applications of Yosida approximations. The book presents optimality of a feedback control for a regulator problem and for equations driven by stochastic vector measures, a periodic control problem under a white noise perturbation, and an optimal control problem for McKean–Vlasov equations.

A serious reader may start at Chapter 2 and pick up the prerequisites on semigroups, Wiener processes, Poisson random measures, and stochastic calculus. A major portion of the book can be understood by adopting a piecemeal approach to reading this chapter. For instance, results on maximal monotone and multivalued operators on Banach spaces and on the central limit theorem by Yosida approximations, etc., can be postponed until multivalued stochastic equations are encountered. A minor criticism of this chapter is that a few results introduced earlier in this chapter depend on notions that are introduced later, e.g., hemicontinuity, demicontinuity, and coercivity. Also, many results are not attributed to the original sources, and the bibliography at the end of the book needs to be augmented carefully.

The next two chapters are devoted to the convergence of Yosida approximations to mild solutions of stochastic evolution equations, with and without jumps, in various contexts and levels of generality. The contexts are often independent of each other, and hence, a reader may skip a few sections to look over a particular type of equation. A thorough discussion of the Itô formula for mild solutions of infinite-dimensional stochastic evolution equations and current research on it (cf. e.g., Ichikawa [4], Da Prato, et al. [2], Albeverio, et al. [1]) would have provided a more compelling argument for the introduction of Yosida approximations.

Chapter 5 studies the stability theory of solutions to illustrate the usefulness and power of Yosida approximations. The results are illustrated by several examples in different contexts. It would have been very helpful if the author had provided more detail on the setup of stochastic equations with jumps and Markov switching which goes back to Skorohod [5, pp. 103–106]. It is likely that the Itô formula for such a stochastic evolution equation and the form of its infinitesimal generator may surprise a reader unacquainted with this level of generality. The final chapter deals with problems in optimal control theory that can be solved by using Yosida approximations.

The book would be interesting and useful to researchers working on stochastic partial differential equations and their applications. The monograph provides a unified treatment of Yosida approximations, stability theory, and stochastic optimal controls for infinite-dimensional stochastic evolution equations. The author has succeeded in collecting a wealth of stochastic evolutions in order to discuss stability and control of their solutions.

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P. Sundar
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana 70803
Email address: `psundar@lsu.edu`