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*Quantum waveguides*, by P. Exner and H. Kovařík, Theoretical and Mathematical Physics, Springer, Cham, 2015, xxii+382 pp., ISBN 978-3-319-18575-0, US$149.99 (hardcover), ISBN 978-3-319-36556-5, US$129.99 (softcover), ISBN 978-3-319-18576-7, US$99.00 (e-book)

It is uncertain when quantum waveguides first made their appearance in the scientific literature. Along with their relatives now known as quantum wires and quantum graphs, they had been considered on occasion since the invention of quantum mechanics under different names and, for various reasons, not necessarily in connection with small electrical structures. The model of an electron traveling in a channel with no interactions other than confinement by the channel's walls was for most of the twentieth century a curiosity without practical implications, and it did not attract wide attention. Macroscopic transport of electrons, like what accounts for the current in the wires in your toaster, is not a simple subject. Even the crude model introduced by Drude in 1900 and described in textbooks such as [25] leads to a type of Boltzmann equation for time-evolving probability distributions by imagining a classical gas of electrons traveling ballistically with a small drift due to the applied electric field, interrupted in some random way by elastic scattering events. Incorporating quantum effects, as first attempted in an ad hoc manner by Sommerfeld, does nothing to simplify matters. Indeed, the justification of key relations of transport theory, such as the Kubo formula, is an active area of research in mathematical physics to the present day.

Transport theory of this kind is the right approach to understand current in macroscopic wires, but a new era dawned in the late 1980s, when it became practical to fabricate electrical devices in the laboratory with widths comparable to the de Broglie wavelength of an electron. On the scale of nanometers the wave nature of the electron dominates, and at modest temperatures and densities it becomes defensible to model the situation with the Schrödinger equation in its one-particle form. If, as in a carbon nanotube, a particle is confined to a thin but very long region, the situation is analogous to the acoustic and electromagnetic waveguides that have been used since the nineteenth century to efficiently carry signals, which were analyzed mathematically by Lord Rayleigh [19,20] as boundary-value problems for the wave equation. (A clear textbook treatment of electromagnetic waveguides may be found in [25]. For a discussion of the complexities of electron transport even on the nanoscale, see [1].) The analogy is even closer when the "quantum waveguide" consists of a region in which the external forces are negligible except near a sharp boundary as defined by a very large potential energy barrier, which in a semiclassical limit is well approximated by a Dirichlet boundary condition. Hence, about three decades ago, quantum waveguides were no longer considered toy models, but viewed as a subject meriting serious analysis.

Quantum mechanics for small numbers of particles relies to a large extent on spectral theory, using eigenvalues and eigenfunctions to understand bound states

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and the continuous spectrum to understand scattering, including resonance behavior [3, 23]. The spectral theory of Schrödinger and Laplace operators has become a well-developed discipline, partly due to the connections with quantum mechanics, but also because it has forged some wonderful connections with geometry, graph theory, and harmonic analysis, which have appeal as pure mathematics.

Exner and Šeba were among the first to understand that while the spectra of quantum waveguides are connected with physics and with geometry, those connections must differ in key respects from what was familiar in an earlier age of spectral analysis. For example, the shapes of quantum wires are essentially tubular neighborhoods of curves, which may be joined into complex networks, and such structures bear scant resemblance to the sorts of domains that had predominated in the earlier literature on whether it is possible to "hear the shape of a drum", à la Mark Kac [14]. Quantum mechanicians and spectral theorists drawn to the subject, most notably a group associated with Exner, found themselves in the gratifying position of well-trained naturalists encountering new species and ecosystems upon arrival at an unexplored shore.

A notable early result of Exner and Šeba, in 1989, was that, under some reasonable assumptions, if a thin but uniformly thick, infinitely long, two-dimensional channel waveguide has any bending whatsoever in some compact region, then it has a bound state (i.e., a discrete eigenvalue below the infimum of the continuous spectrum). The key was to use a coordinate transformation to straighten out the channel while giving rise to an effective potential that is negative, producing a spectrum below the infimum of the essential spectrum (which can be located with Weyl techniques). They did this both in the situation of quantum waveguides with Dirichlet conditions and in the very similar situation of classical electromagnetic waveguides with perfectly conducting edges [8, 9]—work that both caught the eye of mathematical physicists and inspired laboratory experiments; e.g., [5, 6]. Interestingly, although generic bound states in curved electromagnetic waveguides could have been predicted and observed a century earlier, the question appears not to have been asked before [8, 9]. A few years later Goldstone and Jaffe would find a related but more general approach, demonstrating the existence of generic bound states under some circumstances in higher dimensions and higher codimensions [12]. One interesting feature that arrives in dimensions higher than 2 is torsion, which, as first realized by Clark and Bracken [7], leads to a repulsive effective potential that can compete with curvature. What is the nature of the effective potentials induced by geometry, and how does torsion affect the possibility of bound states in tubes and layers? This tricky question inspired some remarkable analysis by several research groups; e.g., [4, 16–18], as well as by the authors of the book under review and their associates. The interplay of curvature and torsion in these models was nicely reviewed a decade ago by Krejčiřík in [15], and the current state of effective Hamiltonian theory has been well treated in a monograph by Wachsmuth and Teufel [24].

In the ensuing two to three decades quantum waveguides and similar models became fashionable in the mathematical physics community, and scores of articles were written by the authors of this monograph and their associates, and by other groups of researchers, rounding up the usual suspects in spectral mathematical physics: the number and location of the eigenvalues, with bounds on various spectral functionals and an understanding of extreme cases; circumstances in which the spectrum is absolutely continuous and wave operators can be constructed; possible gaps in the spectrum; the number and location of resonances; modifications needed to incorporate magnetic fields; and perturbation theory for both the discrete and continuous parts of the spectrum. Although these topics were natural, even inevitable, in the context of modern mathematical physics, each one of them needed serious new analysis to tease out the role of the geometry and the connectedness of the waveguides. Furthermore, some aspects of the physical modeling raised distinctive new questions. Surely, a very thin quantum waveguide should exhibit behavior similar to that of a lower-dimensional structure, whether a quantum wire or a surface, but the limit as the width of the waveguide tends to zero and the dimensionality changes is delicate, and it responds in subtly different ways to the kinds of boundary conditions imposed on the boundary of the waveguide. When the small channels are joined together, and the limit is a quantum graph, the situation becomes quite tricky. For example, if the widths of different channels tend to zero at different rates, a variety of distinct vertex conditions can emerge in the limiting model. The question of how thin waveguide networks are related to the limiting graph was first considered in the 1950s [21], but a full and rigorous treatment of this foundational matter and the related question of categorizing the possible self-adjoint Hamiltonians on quantum graphs took decades to emerge, nearly to the present day. The monograph under review contains the most accessible treatment of vertex conditions, and the history of the subject is recounted in the notes to Chapter 8.

Other topics arising from the physical modeling behind quantum waveguides include waveguides that are not completely isolated but instead are coupled to other waveguides through "windows", and waveguides or quantum wires residing in a larger structure, with respect to which they are "leaky". These models are interesting both for physical reasons and as sources of nice mathematical questions.

The well-written and thorough monograph by Exner and Kovařík contains an excellent treatment of developments in the flourishing subject of quantum waveguides. It lays out the mathematical underpinnings of the subject in an inviting way, covering all of the topics mentioned in this review and more. When paired with Berkolaiko and Kuchment's monograph on quantum graphs [2] it would be a perfect way to prepare a graduate student or researcher wishing to specialize in quantum mechanics on models of nanoscale structures. The monograph is also highly recommended for those with a wider interest in quantum mechanics, since seeing how the concepts and mathematical methods of quantum mechanics need to be adapted to the case of waveguides is an engaging and instructive way to deepen one's understanding of them.

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Evans M. Harrell II
School of Mathematics
Georgia Institute of Technology
Atlanta Georgia 30332-0160
*Email address*: `harrell@math.gatech.edu`