BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, April 2018, Pages 289–293
http://dx.doi.org/10.1090/bull/1606
Article electronically published on December 15, 2017

CORRIGENDUM TO "THE CLASSICAL ARTIN APPROXIMATION THEOREMS"

HERWIG HAUSER

Abstract. The purpose of this note is to clarify and complement various places in the article *The classical Artin approximation theorems*, published in Bull. Amer. Math. Soc. 54 (2017), no. 4, 595–633. A few reasonings in the article may have been difficult to follow or were even problematic. The author is indebted to F. Castro-Jiménez, C. Chiu, R. Ephraim, B. Lamel, S. Perlega, G. Pfister, G. Rond, M. Singer, S. Woblistin, and particularly to an anonymous referee for valuable feedback.

Additions and replacements

Page 599, *line* 7: Add: "For the case of positive characteristic, see [And], [Kie], and [Sche]."

Page 601, *line* 21: Add: "If all components of $f(x,y)$ belong to the ideal $(y_1,\dots,y_m)$ of $\mathbb{K}[[x,y]]$, there always exists the trivial solution $y(x)=0$, and the content of the theorem is void."

Page 601, *line* -3: The argument given in the article for linear equations covers only the case $c=0$, say, without approximation. For arbitary c, see [Ron2, Thm. 3.1] and [Wan] or the argument given below.

"Let $c \geq 1$ be given, and let $h_0,h_1,\dots,h_m \in \mathbb{K}[[x]]^r$. Denote by $U_c \subset \mathbb{K}[x]^m$ the vector space of polynomial vectors of length m in $x=(x_1,\dots,x_n)$ of degree $< c$, and denote by $M_c = (x)^c \cdot (h_1,\dots,h_m) \subseteq \mathbb{K}[[x]]^r$ the $\mathbb{K}[[x]]$-submodule of linear combinations of the vectors h_i with coefficients of order $\geq c$. Choose a monomial order $<_\varepsilon$ on $\mathbb{N}^n \times \{1,\dots,r\}$, let $\mathrm{in}_\varepsilon(M_c)$ be the associated initial module of M_c, and denote by $\mathrm{co}_\varepsilon(M_c)$ the canonical direct monomial complement of $\mathrm{in}_\varepsilon(M_c)$ in $\mathbb{K}[[x]]^r$; see [Gra], [Hir] or [HM1, sec. 5]. The Grauert–Hironaka division theorem for modules then asserts that $\mathrm{co}_\varepsilon(M_c)$ is also a direct complement of M_c itself,

$$M_c \oplus \mathrm{co}_\varepsilon(M_c) = \mathbb{K}[[x]]^r.$$

"Denote by $\psi : \mathbb{K}[[x]]^r \to \mathrm{co}_\varepsilon(M_c)$ the projection on the second summand, corresponding to the remainders of the division of vectors in $\mathbb{K}[[x]]^r$ by the module M_c. It is well known that the coefficients of the expansion of $\psi(u(x))$ are polynomials in the coefficients of the expansions of $u(x) \in \mathbb{K}[[x]]^r$, similar to the classical Weierstrass division theorem. Said differently, ψ is a textile map (see page 603).

"For $u(x) = (u_1(x),\dots,u_m(x)) \in U_c$, consider the vector

$$\widetilde{u}(x) = h_0(x) - \textstyle\sum_{1=1}^m h_i(x) \cdot u_i(x) \in \mathbb{K}[[x]]^r$$

Received by the editors July 17, 2017.

and its remainder $\psi(\widetilde{u}(x)) \in \mathrm{co}_\varepsilon(M_c)$. The set U_c° of vectors $u(x)$ for which $\psi(\widetilde{u}(x)) = 0$, i.e., for which $h_0(x) - \sum_{1=1}^m h_i(x) \cdot u_i(x)$ belongs to M_c, is a closed algebraic subset of U_c, possibly empty. For $u(x)$ outside U_c° the order of $\psi(\widetilde{u}(x))$ with respect to the (x)-adic topology is bounded by the Noetherianity of U_c with respect to the Zariski topology, where the vector space U_c is now considered as an affine algebraic variety. Let o be the maximal value of these orders, and set $e := o + 1$. We claim that this bound satisfies the assertion of the theorem in the linear case.

"So assume that we have an approximate solution $\overline{y}(x)$ of $h_0 = \sum_{i=1}^m h_i \cdot y_i$ modulo $(x)^e$, and denote by $u(x) \in U_c$ the truncation of $\overline{y}(x)$ at degree $c-1$. We distinguish two cases: if $\psi(\widetilde{u}(x)) = 0$, then $h_0(x) - \sum_{1=1}^m h_i(x) \cdot u_i(x) \in M_c$, and there exists a formal power series vector $v(x) = (v_1(x), \ldots, v_m(x)) \in (x)^c \cdot \mathbb{K}[[x]]^m$ so that

$$h_0(x) - \textstyle\sum_{1=1}^m h_i(x) \cdot u_i(x) = \sum_{i=1}^m h_i(x) \cdot v_i(x).$$

Then set $y(x) := u(x) + v(x)$ to get an exact formal solution of $h_0 = \sum_{i=1}^m h_i \cdot y_i$ which satisfies $y(x) \equiv \overline{y}(x)$ modulo $(x)^c$.

"If $\psi(\widetilde{u}(x)) \neq 0$, let o' be its order. We know from the above that $o' < e$. Hence there exists no vector $v(x) = (v_1(x), \ldots, v_m(x)) \in (x)^c \cdot \mathbb{K}[[x]]^m$ so that

$$h_0(x) - \textstyle\sum_{1=1}^m h_i(x) \cdot u_i(x) \equiv \sum_{i=1}^m h_i(x) \cdot v_i(x) \text{ modulo } (x)^e.$$

This is a contradiction to the existence of the approximate solution $\overline{y}(x)$, so the second case does not occur. The assertion is proven."

Page 603, *line* -10: Add: "See [DL1] and [KP] for other presentations."

Page 603, *line* -6: The original formulation of the paragraph is incorrect as it stands. Replace the paragraph starting with "Here is the strategy: …" by the following paragraph:

"Here is the strategy: After suitably modifying the system of equations, one may assume that the components $f_1, \ldots, f_r$ of f generate a complete intersection ideal of height r and that there exists an $(r \times r)$-minor g of the relative Jacobian matrix $\partial_y f$ of f with respect to y for which $g(x, \widehat{y}(x))$ is nonzero and, more specifically, x_n-regular of order d. We will then construct from f a new vector of convergent power series $f' = f'(x', w)$ in the first $n-1$ components $x' = (x_1, \ldots, x_{n-1})$ of the x-variables and in new variables $w = (w_1, \ldots, w_\ell)$ such that the existence of a formal or convergent solution $y(x)$ to $f(x, y) = 0$ for which $g(x, y(x))$ is x_n-regular of order d is *equivalent* to the existence of a formal or convergent solution $w(x')$ of $f'(x', w) = 0$. We will show that these special formal solutions of the first system map to the formal solutions of the latter system, and that the same holds for the convergent solutions (the map can actually be modified so as to give a bijection; see section 9 for an explicit description of this bijection in the case $n = 1$ of one x-variable).

"Now induction applies: by the existence of the formal solution $\widehat{w}(x')$ of $f'(x', w) = 0$ induced by $\widehat{y}(x)$ and the induction hypothesis, the system $f'(x', w) = 0$ admits a convergent solution $\widetilde{w}(x')$, and going backwards we get the required convergent solution $\widetilde{y}(x)$ of the original system $f(x, y(x)) = 0$."

Page 607, *line* -11: The given explanation is somewhat vague. Add to paragraph (a):

"The case $n = 1$, i.e., of a single x-variable, is detailed in section 9: The parametrization is constructed for the stratum of solutions $y(x)$ of $f(x, y) = 0$ for which the order $g(x, y(x))$ of an appropriate minor g of the relative Jacobian matrix $\partial_y f$ is constant and equal to the one of the given formal solution $\widehat{y}(x)$."

Page 607, *paragraph* (b): The ideal I defined in line -9 need not be prime, contrary to what is claimed. This invalidates the given argument. Replace paragraph (b) by:

"(b) The strong approximation theorem I is a weak version of the more precise theorem II. For uncountable algebraically closed ground fields, the result also follows from the strong approximation theorem for textile maps."

Page 608, *paragraph* (c): The second paragraph of (c), starting with "So we may assume ...", is misleading. Replace the entire paragraph (c) by:

"(c) The proof of the strong approximation theorem II requires a double induction, one decreasing on the number n of x-variables and, subordinate to this induction, one increasing on the height s of the ideal I. We will only indicate here the main ideas. See [BDLv] for a model-theoretic proof using ultraproducts and [Wa] for a proof along the lines of the analytic version of the approximation theorem.

"One first reduces again to prime ideals. So let I be the ideal generated by the components $f_1, \dots, f_r$ of f. Choose an irredundant primary decomposition $I = I_1 \cap \cdots \cap I_t$, and let $J_i = \sqrt{I_i}$ be the associated prime ideals. As $\mathbb{K}[[x]]$ is Noetherian, there exists an integer u such that $J_i^u \subset I_i$ for all i. Let e_i be the bound associated to J_i by the theorem in case of prime ideals. Then $e = u \cdot (e_1 + \cdots + e_t)$ will work for I; cf. [Wa, proof of Lemma 5, p. 133]. Namely,

$$J_1^u \cdots J_t^u \subset I_1 \cdots I_t \subset I_1 \cap \cdots \cap I_t = I,$$

so that any approximate solution $\overline{y}(x)$ for I up to degree e is also an approximate solution of some I_i up to degree e_i, for some i. By assumption, I_i then admits an exact formal solution $\widehat{y}(x)$ with $\widehat{y}(x) \equiv \overline{y}(x)$ modulo $(x)^c$. From $I \subset I_i$ it follows that $\widehat{y}(x)$ is also an exact formal solution for I.

"So we may assume that I is prime. Let s be its height and choose again an $(s \times s)$-minor g of the Jacobian matrix so that $g \notin I$. Let $I' = I + (g)$. It has height $> s$, so that induction applies to it. Denote by e' the respective value for I' and c. At this point, one distinguishes two cases: if $g(x, \overline{y}(x)) \equiv 0$ modulo $(x)^{e'}$ holds for all approximate solutions $\overline{y}(x)$ of I up to degree e', one may set $e = e'$ and get the required assertion. In the other case, the argument becomes much more involved: according to the orders $d = \operatorname{ord} g(x, \overline{y}(x)) < e'$ of approximate solutions $\overline{y}(x)$ one essentially repeats the construction of an associated system of equations in one variable less as we have seen it in the proof of the analytic approximation theorem. From this one is able to find a suitable bound e for the original system. We refer to [Wa], section 2, for the details."

Page 609, *line* 12: Add: "For a critical discussion of Hermann's work, see [Sei], [Sto]."

Page 611, *second paragraph of item* (e): We add a note: "To prove the existence of the sequence c_k the continuity of the map G with respect to the Krull topology is actually not used: As the ith coefficient of $G(y)$ only depends (polynomially) on finitely many coefficients of y, we see that it equals the ith coefficient of $G(\overline{y})$ as soon as y and $\overline{y}$ coincide up to sufficiently high degree."

Page 615, *line* −15: Add: "In contrast to what is said, algebraic series *are* closed under composition (= substitution of the variables): If $h(y_1, \ldots, y_m)$ and $g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)$ are algebraic series, so is $h(g_1(x), \ldots, g_m(x))$; see [Rui, Prop. 5.3, page 114] or [Bos, lecture II, Thm. p. 15]. In the univariate case $m = n = 1$, if $P(x,t)$ and $Q(x,t)$ are the minimal polynomials of $h(x)$ and $g(x)$, then $h(g(x))$ is a root of the resultant $\operatorname{res}_z(P(x,z), Q(z,t))$."

Page 623, *line* 4, *preceding the first sentence*: Add: "For reduced germs, the theorem has originally been proven by Ephraim [Eph, Th. 03]."

Page 625, *lines* 24–25: Replace: "This has been globalized to a certain extent by Bouthier and Kazhdan [BK]" by "Recently, the statement of the theorem has been reproven independently by Bouthier and Kazhdan [BK, Thme. 1.29]."

Typographical errors

Page 597, *Example*: Replace "$F_0 = y_0$" by "$F_0 = y_0^3$"; and replace "$F_3 = y_1^3 + 3y_1y_2 - 2z_0z_2 - z_1^2$" by "$F_3 = y_1^3 + 6y_0y_1y_2 + 3y_0^2y_3 - 2z_0z_2 - z_1^2$".

Page 597, *the paragraph following Example*: Replace from "We may take $y_1 = 1$ and ..." up to "Continuing like this, ..." by "As $z_0 = 0$, we have $F_3 = y_1^3 - z_1^2$, and we may take $y_1 = 1$ and $z_1 = 1$. This is a permissible choice, though not unique. We could also take $y_1 = 100$ and $z_1 = 1000$. Continuing like this, ..."

Page 601, *strong approximation theorems I & II*: Replace "$(x) \cdot \mathbb{K}[[x]]^m$" by "$(x) \cdot \mathbb{K}[[x]]^r$".

Page 603, *strong approximation theorem for textile maps*: Replace "Let $G : \mathbb{K}[[x]]^m \to \mathbb{K}[[z]]^q$ be a textile map" by "Let $G : \mathbb{K}[[x]]^m \to \mathbb{K}[[x]]^r$ be a textile map".

Page 607, *line* −8: Replace "ideal of convergent power series $h(x,y)$ for which ..." by "ideal of formal power series $h(x,y)$ for which ...".

Page 611, *line* 16: Replace "$\mathbb{A}_{\mathbb{K}}^{mnk}$" by "$\mathbb{A}_{\mathbb{K}}^{mn_k}$ with $n_k = \binom{n+k-1}{n}$".

Additional and updated references

[And] Michel André, *Artin's theorem on the solutions of analytic equations in positive characteristic*, Manuscripta Math. **15** (1975), no. 4, 341–347. MR0379493

[BK] A. Bouthier and D. Kazhdan, *Faisceaux pervers sur les espaces d'arcs*, 2017. arXiv:1509.02203v5

[Kie] Reinhardt Kiehl, *Ausgezeichnete Ringe in der nichtarchimedischen analytischen Geometrie.* (German), J. Reine Angew. Math. **234** (1969), 89–98. MR0243126

[KP] Herbert Kurke and Gerhard Pfister, *Weierstrass categories and the property of approximation*, Math. Nachr. **109** (1982), 249–280. MR705908

[Sche] Klaus-Peter Schemmel, *Eine notwendige und hinreichende Bedingung für die Approximationseigenschaft analytischer Potenzreihenringe über einem Körper beliebiger Charakteristik* (German), Rev. Roumaine Math. Pures Appl. **27** (1982), no. 8, 875–884. MR683549

[Sei] A. Seidenberg, *Constructions in algebra*, Trans. Amer. Math. Soc. **197** (1974), 273–313. MR0349648

[Sto] Gabriel Stolzenberg, *Constructive normalization of an algebraic variety*, Bull. Amer. Math. Soc. **74** (1968), 595–599. MR0224602

[Wan] Ti Wang, *A stratification given by Artin-Rees estimates*, Canad. J. Math. **44** (1992), no. 1, 194–205. MR1152675

Faculty of Mathematics, University of Vienna, Austria
Email address: herwig.hauser@univie.ac.at