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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
FELIX SCHLENK

MR0809718 (87j:53053) 53C15; 32F25, 53C57, 57R15

Gromov, M.

Pseudo holomorphic curves in symplectic manifolds.

Inventiones Mathematicae **82** (1985), *no.* 2, 307–347.

The paper under review opens a new effective approach to fundamental problems of symplectic topology. Let (M,ω) be a symplectic manifold. An almost complex structure J on M is said by the author to be tamed by ω if $\omega(x,Jx)>0$ for all nonzero tangent vectors. Almost complex structures tamed by the given symplectic form are sections of a fiber bundle with a contractible fiber. In particular they will always exist. The author's theory shows that manifolds with such structures have (like Kähler complex analytic manifolds) many globally defined (pseudo)holomorphic curves (or J-curves), which leads to many deep results in the geometry and the topology of contact and symplectic manifolds. The following theorems illustrate the character of numerous results of the paper. Let S^2 be the 2-sphere with the area form ω_1 with $\int_{S^2}\omega_1=A_1$ and let V_2 be a closed manifold of dimension $2(n-1)$ with a symplectic form ω such that $\int_{S^2}\omega_2=kA_1$ for every smoothly mapped sphere $S^2\to V$ for some integer $k=k(S^2\to V)$. Theorem (2.3.C): Let J be a C^∞-smooth almost complex structure on $V=S^2\times V_2$ tamed by the symplectic form $\omega_1\oplus\omega_2$. Then there exists a (possibly singular and nonunique) rational (i.e., diffeomorphic to S^2) J-curve $C=C_v\subset V$ which contains a given point $v\in V$ and which is homologous to the sphere $S^2\times v_2\subset V$, $v_2\in V_2$. If $n=2$ and V_2 is not diffeomorphic to S^2, or $k>1$, then C is regular and unique. If V_2 is diffeomorphic to S^2 and $k=1$ then there exists a connected regular J-curve C in V which represents the homology class $p[S^2]+q[V_2]\in H^2(V;\mathbf{Z})$ for arbitrary nonnegative integers p and q and which has $\mathrm{genus}(C)=pq-p-q+2$. In fact, these curves C form a smooth manifold $M=M_{pq}(J)$ of dimension $2(pq+p+q)$. Corollary (0.3.A): Consider a symplectic diffeomorphism of the open round ball $B(R)\subset\mathbf{R}^{2n}$ onto an open subset $V'\subset\mathbf{R}^{2n}$ which is contained in the ε-neighborhood of the symplectic subspace $\mathbf{R}^{2n-2}\subset\mathbf{R}^{2n}$. Then R satisfies the inequality $R\le\varepsilon$.

The next result shows the uniqueness of symplectic structure on $\mathbf{R}^4$. Theorem (0.3.C): Let an open manifold (V,ω) be symplectically diffeomorphic to $(\mathbf{R}^4,\omega_0=dx_1\wedge dy_1+dx_2\wedge dy_2)$ at infinity. If the Hurewicz homomorphism $\pi_2(V)\to H_2(V;\mathbf{R})$ vanishes, then (V,ω) is symplectically diffeomorphic to $(\mathbf{R}^4,\omega_0)$. Now consider $\mathbf{C}^n$ with standard complex and symplectic structures. Theorem (0.4.A_2): For an arbitrary closed C^∞ smooth Lagrange submanifold $W\subset\mathbf{C}^n$ there exists a nonconstant holomorphic map $f\,(D^2,\partial D^2)\to(\mathbf{C}^n,W)$. It follows that the relative class $[\omega_0]\in H^2(\mathbf{C}^n,W;\mathbf{R})$ is nonzero. Corollary (0.4.A_2'): There exists a symplectic structure ω on $\mathbf{R}^{2n}$ for all $n\ge 2$ which admits no symplectic embedding into $(\mathbf{R}^{2n}=\mathbf{C}^n,\omega_0)$.

The author successfully applies his theory to prove new fixed point theorems for exact symplectic diffeomorphisms and to get many deep results in contact topology and between them, e.g., D. Bennequin's theorem [*Troisième rencontre de géométrie* *du Schnepfenried, Vol. 1* (Schnepfenried, 1982), 87–161, Astérisque, 107-108, Soc. Math. France, 1983; MR0753131] and some of its higher-dimensional analogues. One of the main tools of the theory is the compactness theorem for the space of pseudoholomorphic curves. The author introduces the notion of weak convergence of pseudoholomorphic curves to a "cusp-curve" and proves the following theorem (1.5.B): Let V be a compact manifold with almost complex structure J and Riemannian metric μ. Let C_j be a sequence of closed J-curves of fixed genus in (V, J, μ). If the areas $\mathrm{Area}_\mu C_j$ are uniformly bounded then some subsequence weakly converges to a cusp-curve $\overline{C}$ in V.

Compare 1.5.B with Bishop's compactness theorem for analytic sets [see E. Bishop, Michigan Math. J. **11** (1964), 289–304; MR0168801].

Yakov Eliashberg

From MathSciNet, January 2018

MR1438190 (98h:53045) 53C15; 53C40, 53C65, 57R15, 58F05

Donaldson, S. K

Symplectic submanifolds and almost-complex geometry.

Journal of Differential Geometry **44** (1996), *no.* 4, 666–705.

This important paper develops a general procedure for producing symplectic submanifolds of any even codimension within a given compact symplectic manifold (V, ω) of dimension $2n$, $n \geq 2$. The main construction of the paper is the following deep existence theorem for codimension 2 symplectic submanifolds. Assume that the class $[\omega/2\pi] \in H^2(V; \mathbf{R})$ is "rational", i.e. there is a lift of $[\omega/2\pi]$ to an integral class $h \in H^2(V; \mathbf{Z})$. Then for sufficiently large integers k, the Poincaré dual of kh in $H_{2n-2}(V; \mathbf{Z})$ may be represented by a symplectic submanifold $W \subset V$.

In order to obtain the symplectic submanifold W the author first endows V with a compatible almost complex structure and considers a complex line bundle $L \to V$ with first Chern class $c_1(L) = [\omega/2\pi]$. In the integrable case, when V is a complex manifold, since L is a positive line bundle one may apply the Kodaira embedding theorem and conclude that for k sufficiently large, $L^{\otimes k}$ gives rise to a projective embedding of V in $\mathbf{CP}^{\mathbf{N}}$. If $s_k \colon V \to L^{\otimes k}$ is a holomorphic section which is transverse to the zero section then $W_k = \{x \in V\colon\ s_k(x) = 0\}$ is a complex submanifold and hence a symplectic submanifold (equivalently, W is given by a hyperplane section of $V \subset \mathbf{CP}^{\mathbf{N}}$). In the nonintegrable case this result is significantly more difficult. In particular, holomorphic sections will not exist in general. One must instead try to find sections, transverse to the zero section, which are as near to holomorphic as possible. The author shows that there is a transverse section s of $L^{\otimes k}$ which satisfies $|\overline{\partial} s| < (C/\sqrt{k})|\partial s|$ on the zero set of s, where ∂s and $\overline{\partial} s$ are the complex linear and antilinear parts of the derivative ∇s and C is a positive constant. It is easy to see that this inequality ensures that the submanifold $W = \{x \in V\colon\ s(x) = 0\}$ is symplectic. Proving the existence of a transverse section which satisfies this inequality is a delicate analytic task which makes up the core of the paper. Roughly speaking, the proof consists of two parts. The first is an analytic construction of "approximately holomorphic" sections s with $\overline{\partial} s$ small. The second, more difficult, part is to show that among these sections one can find ones where ∂s is not small on the intersection of s with the zero section. The local part of the proof of this "quantitative transversality" result is accomplished by adapting a technique from real algebraic geometry developed by Y. Yomdin [Math. Ann. **264** (1983), no. 4, 495–515; MR0716263].

For large values of k these symplectic submanifolds are “approximately pseudo-holomorphic”. This is the key point that allows the author to remove the condition that the class $[\omega/2\pi]$ be rational. If one begins with an arbitrary compact symplectic manifold (V,ω) then one can find an arbitrarily close symplectic structure ω' on V which is rational. If J' is an almost complex structure which is compatible with ω' then the approximately J'-pseudo-holomorphic submanifolds will also be symplectic with respect to ω. By replacing V with W and iterating the construction one produces submanifolds of arbitrary even codimension.

In the final section the author establishes a number of related geometric results. He proves an asymptotic result which shows that the submanifolds W are quite complicated, essentially filling out all of V as $k\to\infty$. In particular, when viewed as currents, the sequence $k^{-1}W_k$ converges to a multiple of the symplectic form ω.

By adapting the main construction to the integrable case, the author is able to obtain bounds on the first fundamental form and curvature of complex hypersurfaces of high degree (representing the Poincaré dual of $k[\omega]$ for large k) in a Kähler manifold. These estimates are optimal in terms of their dependence on k. The author raises the interesting problem of trying to find the optimal constant when $V=\mathbf{C}\mathrm{P}^2$; in other words one would like to understand which curves of high degree are the “smoothest” (or rather “flattest”) among all curves representing the same homology class.

In addition the author proves an analogue of the Lefschetz hyperplane theorem which states that for k sufficiently large the inclusion of W_k in V induces an isomorphism on the homotopy groups π_p for $p\le n-2$ and a surjection on π_{n-2}.

Prior to the theory developed here, general methods for producing symplectic submanifolds were available only in either codimension $d\ge 4$ or dimension 4 (i.e. $n=2$). In the former case, under certain natural topological assumptions, M. Gromov's h-principle may be used to produce symplectic submanifolds of codimension $d\ge 4$ [see *Partial differential relations*, Springer, Berlin, 1986; MR0864505]. In the latter case one has the theory of pseudo-holomorphic curves, also introduced by Gromov. This has been an extremely important tool in the study of symplectic four-manifolds and has recently seen remarkable applications, primarily through the work of C. H. Taubes [see, e.g., J. Differential Geom. **44** (1996), no. 4, 818–893; MR1438194]. The paper under review presents the only general existence result known for codimension 2 symplectic submanifolds in dimensions greater than 4. It is clearly the beginning of a very rich theory.

Daniel Pollack

From MathSciNet, January 2018

MR1804164 (2001k:53169) 53D35; 57R17

Auroux, Denis; Katzarkov, Ludmil

Branched coverings of $\mathbf{C}\mathrm{P}^2$ and invariants of symplectic 4-manifolds.

Inventiones Mathematicae **142** (2000), *no.* 3, 631–673.

This interesting paper develops in a very concrete fashion the analogue for symplectic four-manifolds of a broad theory of “braid monodromies” of algebraic surfaces. Every algebraic surface may be described as a branched cover of the projective plane. Pioneering work of B. G. Moishezon [in *Combinatorial methods in topology* *and algebraic geometry (Rochester, N.Y., 1982)*, 311–344, Contemp. Math., 44, Amer. Math. Soc., Providence, RI, 1985; MR0813122] and of M. Teicher [in *Algebraic geometry—Santa Cruz 1995*, 127–150, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997; MR1492521] showed that certain combinatorial invariants of this description were essentially computable, and could be used for instance to distinguish connected components of moduli spaces of surfaces of general type (an idea attributed by Moishezon to Chakiris). The computations themselves are very hard, but the formal picture of describing a complex surface via the monodromy representation of a branched cover is elementary and appealing.

Work of S. K. Donaldson [J. Differential Geom. **44** (1996), no. 4, 666–705; MR1438190] in symplectic geometry, extended by Auroux [Invent. Math. **139** (2000), no. 3, 551–602; MR1738061], shows that symplectic four-manifolds may also be written as branched covers of the projective plane. One can therefore hope to produce similar combinatorial invariants. Here there are two caveats; the branched covers become symplectically canonical only at arbitrarily large degree k, and the branch loci in the projective plane may have nodes of *negative* self-intersection. The latter, for instance, means that the Moishezon-Teicher arguments do not immediately imply that the fundamental group of the complement of the branch curve is independent of choices in the construction; the asymptotic uniqueness means that you are faced with an infinite sequence of branch curves in any case.

The paper under review is rather technical and applications are deferred for sequel papers. It is worth stressing, therefore, that over the last two years the same authors (together with M. Yotov) have brought these methods much further, and it seems very likely that the foundations laid here lead to symplectic invariants that distinguish pairs of simply-connected symplectic four-manifolds which are homeomorphic and have the same Seiberg-Witten invariants. It would follow that, for differential topology at least, these invariants would be more sensitive than anything known to date. It is also worth pointing out, however, that the strategies of these later proofs have developed in directions beyond those suggested in the present paper, and the invariants and examples described here are not those to which the sequels return.

Returning to the task at hand, Auroux proved in [op. cit.] that symplectic four-manifolds cover the projective plane. To obtain invariants from these covers, one first needs to arrange the branch loci into generic and computationally tractable form, and the main goal of this paper is to do just this. The notion of an approximately holomorphic covering is refined to a quasiholomorphic covering (Definition 1) in which the branch curve is forced to lie so that projection from a distinguished point of $\mathbf{P}^2$ to a complex line maps the branch curve generically onto the line, leading to a distinguished presentation of the fundamental group of its complement. Most of the work of the paper returns to the "estimated transversality" arguments of Auroux's original construction, to show that these can be adapted to give this additional regularity. Similarly, the old one-real-parameter construction goes over to give the obvious uniqueness statement. The converse is much easier (and probably standard in the field)—given all the combinatorial data, one can associate a symplectic manifold, well-defined up to symplectomorphism. In principle, then, we have a classification, and the question of applicability arises.

Note that the discussion after Corollary 2 of Section 4 remains mysterious, and indeed the applications of later papers do not rely on removing negative nodes but rather on measuring their effect.

The last sections of the paper develop two welcome digressions. In the first, the authors show in detail how to construct Lefschetz pencil monodromy data from that of branched coverings and braid factorisations. That this is possible is unsurprising (nets of sections of a line bundle clearly contain all the information of a pencil), but the detailed construction illuminates two factors. First, the construction is universal and effected by a homomorphism from a natural subgroup of the braid group to the mapping class group (cf. Remark 7). Second, this homomorphism is trivial on braid elements coming from nodes and cusps of the branch curve. This leads to much simpler formulae for Lefschetz pencils and their monodromy (positive relations), but also suggests that the invariants the authors are chasing—fundamental groups of complements of branch curves, for instance—will remain invisible in the smaller linear system, even though this is usually already enough to rebuild the symplectic manifold to symplectomorphism.

Towards the end of the paper, the authors illustrate their arguments and computations for the cubic surface in $\mathbf{P}^3$. Although the material is by its nature rather dense, the illustration is well chosen and—bearing this example in mind—the earlier sections of the paper became less intimidating, at least to this reviewer.

Ivan Smith

From MathSciNet, January 2018

MR1844078 (2002g:53153) 53D35; 32Q28, 32Q65

Biran, P.

Lagrangian barriers and symplectic embeddings.

Geometric and Functional Analysis **11** (2001), *no.* 3, 407–646.

A classical problem in symplectic topology is to determine an optimal bound on the size of a ball which can be symplectically embedded into a given symplectic manifold (M, Ω). In fact it was one of Gromov's first applications of pseudoholomorphic curves to compute what is now called Gromov width for a variety of symplectic manifolds [M. L. Gromov, Invent. Math. **82** (1985), no. 2, 307–347; MR0809718]. In particular he determined that for a product $(M, \omega) \times \mathbf{C}\mathrm{P}^1$ the radius r of any such ball is subject to the condition that the corresponding area πr^2 of the disk does not exceed the area of $\{*\} \times \mathbf{C}\mathrm{P}^1$, provided that M does not admit a J-holomorphic sphere of area less than the area of $\{*\} \times \mathbf{C}\mathrm{P}^1$.

The main subject of the paper is a decomposition of Kähler manifolds (which should also be valid for general symplectic manifolds) into a Stein manifold and a disk bundle whose Chern class is represented by the Kähler form (symplectic form) on the base equipped with a canonical symplectic structure. The Stein manifold is described by its skeleton—a finite CW-complex to which it contracts under the (positive) gradient flow of a plurisubharmonic function on it.

More precisely, let Ω be an integer homology class. Assume there is a smooth and reduced hypersurface $\Sigma \subset M$ whose Poincarédual is $k\Omega$ for some $k \in \mathbf{N}$. Σ is called a polarization of the Kähler manifold; the complete set of data is denoted by $\mathcal{P}$. Since its Ω-area is positive, any complex curve in M has to intersect Σ. Hence the complement of Σ is Stein.

In fact, let Φ be a section of the holomorphic line bundle $\mathcal{L}$ having Σ as its divisor, with Σ as its regular zero set. Fix a Hermitian metric $\|\cdot\|$ on it such that the connection ∇, compatible with this metric and the holomorphic structure, has curvature $R^\nabla = 2\pi i k\Omega$. The pointwise length $\phi_{\mathcal{P}} = -(1/4\pi)\log\|\Phi_{\mathcal{P}}\|^2$ is an exhausting plurisubharmonic function of $M\backslash\Sigma$. It is independent of the additional choices apart from the data describing the polarization. Now $-dd^c\phi_{\mathcal{P}} = k\Omega$ and $\|\Phi_{\mathcal{P}}\|^2$ has no critical points away from Σ in a sufficiently small neighborhood. Then $\Delta_{\mathcal{P}} \subset\subset M\backslash\Sigma$ is defined to be the union of the unstable submanifolds of critical points in $M\backslash\Sigma$ of the gradient flow of $\phi_{\mathcal{P}}$ with respect to the Kähler metric. If the critical point is nondegenerate then its unstable submanifold will be isotropic since $\phi_{\mathcal{P}}$ is plurisubharmonic (Lemma 8.1.A). Hence their (real) dimension is not bigger than the complex dimension of M. $\Delta_{\mathcal{P}}$ is called the skeleton of the polarization.

$M\backslash\Delta_{\mathcal{P}}$ is shown to be symplectomorphic to a standard symplectic disk bundle (E,ω_0) over Σ, modeled on the normal bundle N_Σ of Σ in M, whose fibres have area $1/k$.

If $\phi_{\mathcal{P}}$ is Morse then the skeleton has the structure of a cellular subspace (Definition 2.6.B) which lacks two properties of CW-complexes: the attaching of the cells along their boundaries to lower-dimensional strata and the strictly decreasing dimensions of the strata. However, there is a CW-complex with a homotopy equivalence to the skeleton which preserves dimensions, such that its complement is again symplectomorphic to (E,ω_0) (Theorem 2.6.C). One main contribution to the proof is a plurisubharmonic version of the Kupka-Smale theorem: One can perturb $\phi_{\mathcal{P}}$ in a compact neighborhood of the set of critical points in $M\backslash\Sigma$ to a Morse function ϕ with the following properties: (1) It is, of course, still plurisubharmonic and coincides with $\phi_{\mathcal{P}}$ in a neighborhood of Σ. (2) Its gradient flow with respect to the Kähler metric with Kähler form $\omega_\phi = -dd^c\phi$ is Morse-Smale. Then the union of unstable manifolds of the critical points Δ_ϕ will be an isotropic CW-complex. On the other hand ω_ϕ is isotopic to the original symplectic form Ω, and due to Moser's argument they are symplectomorphic. The image of Δ_ϕ under this symplectomorphism will be the replacement for $\Delta_{\mathcal{P}}$ we were looking for. Biran and K. Cieliebak [Israel J. Math. **127** (2002), 221–244] studied the case when the dimension of Δ_ϕ is strictly smaller than half the dimension of M.

The author computes the skeleton in several examples of polarizations. The main application of the decomposition is that the Gromov width of $M\backslash\Delta$ is considerably smaller than the Gromov width of M itself. This follows from the following principle: Whatever can be symplectically embedded into (M,Ω) but not into (E,ω_0) must intersect Δ.

Biran calls this phenomenon Lagrangian barriers. With assumptions on (M,Ω) which are basically the same as those to ensure the existence of Gromov-Witten invariants by perturbing the almost complex structure, he concludes that every symplectic ball of radius λ, $B(\lambda)$, with $\lambda^2 > 1/\pi k$ must intersect Δ. Algebraic geometric arguments, on the other hand, show that this inequality is sharp if the line bundle defined by Σ and restricted to it, $\mathcal{O}(\Sigma)|_\Sigma$, is globally generated by holomorphic sections, which is the case provided k is sufficiently large (Theorem 4.A).

It is pointed out in the paper that the results should be correct in a much more general context. First of all, the conditions on (M,Ω) could probably be removed using the more general multi-valued perturbations of the data invented in the context of Gromov-Witten invariants. Second, if (M,Ω) is symplectic but not necessarily Kähler then, due to S. K. Donaldson [J. Differential Geom. **44** (1996), no. 4, 666–705; MR1438190], there is a hypersurface Poincarédual to $[N\Omega]$ if Ω was a rational class to begin with and N is sufficiently large. However, the algebraic methods used in the proofs (in particular the ampleness of the line bundle $\mathcal{L}$) are to be adopted or replaced by techniques appropriate to this more general setting.

Klaus Mohnke

From MathSciNet, January 2018

MR2231465 (2007b:53178) 53D35; 53D12, 53D40

Biran, P.

Lagrangian non-intersections.

Geometric and Functional Analysis **16** (2006), *no.* 2, 279–326.

A recurring theme in symplectic topology since its very beginnings has been the fact that the topology of Lagrangian submanifolds of symplectic manifolds is often more constrained than that of smooth submanifolds of the same dimension. In this paper, the author uses some of his previous work (especially [P. Biran, Geom. Funct. Anal. **11** (2001), no. 3, 407–464; MR1844078] and [P. Biran and K. Cieliebak, Israel J. Math. **127** (2002), 221–244; MR1900700]) to develop new methods that reveal further examples of this phenomenon. For instance, it is shown that if X is a $2m$-dimensional closed symplectic manifold such that $\pi_2(X) = 0$ and if $\mathbb{C}\mathrm{P}^n \times X$ contains a Lagrangian sphere, then m is an odd multiple of $n+1$; additionally, when $m = n+1$ the only simply-connected Lagrangian submanifolds L of $\mathbb{C}\mathrm{P}^n \times X$ satisfy $H^*(L; Z_2) \cong H^*(S^{2n+1}; Z_2)$. Another result proven in this paper states that if Q is the standard quadric hypersurface of $\mathbb{C}\mathrm{P}^n$ with $n \geq 3$, and $\Lambda \subset Q$ is its real locus, then any Lagrangian submanifold $L \subset Q$ with vanishing first homology must intersect Λ. Several other results in a similar vein are proven as well.

The proofs of these results proceed by first translating the question of the existence of the Lagrangian submanifold L into that of the existence of a Hamiltonianly displaceable Lagrangian submanifold Γ in a certain Stein manifold (which necessarily has vanishing Floer homology), and then using a spectral sequence similar to that constructed in [Y.-G. Oh, Internat. Math. Res. Notices **1996**, no. 7, 305–346; MR1389956] to see that the Floer homology of Γ cannot vanish unless certain constraints on the singular cohomology of L are satisfied. To describe the general setup, assume that (M, ω, J) is a Kähler manifold containing a smooth hypersurface Σ Poincaré dual to a multiple of $[\omega]$ (in the first theorem mentioned in the previous paragraph, M is $\mathbb{C}\mathrm{P}^{n+1}$ and Σ is a hyperplane). Then $M \backslash \Sigma$ is Stein, with exhausting plurisubharmonic function ϕ given by a multiple of the logarithm of the norm-squared of a section of a line bundle vanishing along Σ. Similar to the situation in [P. Biran, op. cit.], it is seen that M decomposes symplectically as the union of a standard disc bundle over Σ with an isotropic CW complex constructed from the stable manifolds of a Morse-Bott perturbation of ϕ. Let P be the unit circle bundle in this disc bundle. Over any Lagrangian L in M (or more generally in $M \times X$ for any other symplectic manifold X), one considers the preimage $\Gamma_L \subset P \times X$, which is a Lagrangian submanifold of $(M\backslash\Sigma) \times X$. Now if $M\backslash\Sigma$ is subcritical (i.e., the isotropic CW complex mentioned earlier has no cells of dimension $\frac{1}{2}\dim M$; this holds for instance if $M = \mathbb{C}\mathrm{P}^{n+1}$ and Σ is a hyperplane), then a result from [P. Biran and K. Cieliebak, op. cit.] shows that Γ_L can be displaced from itself by a Hamiltonian symplectomorphism, so that the Floer homology $HF(\Gamma_L, \Gamma_L)$ vanishes assuming it is defined. Under a somewhat more general assumption on the critical submanifolds of the Morse-Bott exhausting plurisubharmonic function on $M \backslash \Sigma$, which is satisfied when $M = \mathbb{C}\mathrm{P}^{n+1}$ and Σ is a hypersurface of any degree, the author proves a more general analogue of the above-mentioned result of [P. Biran and K. Cieliebak, op. cit.] which shows that there is a union $\Lambda \subset \Sigma$ of immersed Lagrangian spheres with the property that if $L \cap \Lambda = \varnothing$ then Γ_L can be displaced from itself by a Hamiltonian symplectomorphism. Once this is proven, the main results become relatively simple algebraic consequences of (an analogue of) the Oh spectral sequence, which has E_1 term expressed in terms of $H^*(\Gamma_L; Z_2)$ and E_∞ term consisting of copies of $HF(\Gamma_L, \Gamma_L)$. In each case, the assumption on L (e.g., that it is a Lagrangian submanifold of the quadric Q which misses the real locus Λ) forces $HF(\Gamma_L, \Gamma_L)$ to be well-defined and zero, which implies that all of the elements of $H^*(\Gamma_L; Z_2)$ must be killed by the higher differentials of the spectral sequence; the grading properties of this spectral sequence together with the Gysin sequence for the circle bundle $\Gamma_L \to L$ then lead to the desired conclusion about the singular cohomology of L (e.g., that it must be nonvanishing in degree 1).

While the results of this paper are heavily dependent on previous work, summaries of that work are provided which serve to make this paper quite readable. The main results here, while generally confined to somewhat special cases, are rather striking, and it would be interesting to know how far they can be generalized.

Michael J. Usher

From MathSciNet, January 2018

MR2499436 (2010b:53155) 53D35; 57R17

McDuff, Dusa

Symplectic embeddings of 4-dimensional ellipsoids.

Journal of Topology **2** (2009), *no.* 1, 1–22.

Symplectic embedding problems have played a prominent role in symplectic topology since the time of M. L. Gromov's Nonsqueezing Theorem. This paper shows how to convert the problem of symplectically embedding one 4-dimensional rational ellipsoid into another to a problem of embedding disjoint unions of balls into $\mathbb{C}\mathrm{P}^2$, by using a new way to desingularize orbifold blow-ups of weighted projective spaces. More precisely, the author shows that the ellipsoid $E(1,k)$ (where $k \in \mathbb{N}$ is the ratio of the area of the major axis to that of the minor axis) embeds in the open ball $\overset{\circ}{\to} B(\mu)$ if and only if k disjoint (closed) balls $B(1)$ embed in $\overset{\circ}{\to} B(\mu)$. Moreover, it is shown that the general ellipsoid embedding problem is equivalent to the symplectic packing problem for k balls with weights $w := (w_1, \dots, w_k)$, that is, the problem of embedding k disjoint (closed) balls $B(w_1), \dots, B(w_k)$ into the open ball $\overset{\circ}{\to} B(1)$. Therefore these questions can then be solved by previous work of Gromov [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718], P. Biran [Geom. Funct. Anal. **7** (1997), no. 3, 420–437; MR1466333] and D. McDuff and L. Polterovich [Invent. Math. **115** (1994), no. 3, 405–434; MR1262938], by converting them into questions about the existence of symplectic forms on the k-fold blow-up of $\mathbb{C}\mathrm{P}^2$. As a consequence, the author shows that the ball may be fully filled by the ellipsoid $E(1,k)$, for $k = 1, 4$ and all $k \geq 9$. Another important corollary answers negatively to a question posed by K. Cieliebak et al. [in *Dynamics, ergodic theory, and geometry*, 1–44, Cambridge Univ. Press, Cambridge, 2007; MR2369441], where they asked if the volume and the Ekeland-Hofer capacities are the only obstructions to embedding one open ellipsoid into another.

Section 2 is devoted to the proof of the particular case of embedding ellipsoids of the form $E(1,k)$, where $k \in \mathbb{N}$, into balls, while the general case is proved in Section 3.

The main idea of the proof is to cut the ellipsoid into balls via toric models. The toric picture of a rational ellipsoid is a singular triangle, that is, the image of the ellipsoid under the moment map of the obvious T^2 action in $\mathbb{C}^2$ is a singular triangle which coincides with the moment polytope of a weighted projective plane. (The intersection of two successive edges of a polytope in $\mathbb{R}^2$ is a singular point if the matrix whose rows are outward normals to these edges has determinant $\neq \pm 1$.) The idea is then to show that these triangles can be cut into standard triangles of different sizes. Moreover, this decomposition also corresponds to a joint resolution of the two singularities of the toric variety corresponding to the complement of the triangle in the positive quadrant. The cuts needed to resolve the singularities in this polytope are parallel to the cuts that decompose the singular triangle into standard triangles.

This problem also turns out to have very interesting and unexpected relations to the properties of continued fractions, as explained by the author in Section 3.1.

{For further information pertaining to this item see [D. McDuff, J. Topol. **8** (2015), no. 4, 1119–1122; MR3431670].}

Sílvia R. Anjos
From MathSciNet, January 2018

MR2838266 53D42
Hutchings, Michael
Quantitative embedded contact homology.
Journal of Differential Geometry **88** (2011), *no.* 2, 231–266.

In the article under review, the author defines new obstructions to symplectically embedding one 4-dimensional Liouville domain into another via embedded contact homology (ECH), which was also defined by the author [J. Eur. Math. Soc. (JEMS) **4** (2002), no. 4, 313–361; MR1941088]. A Liouville domain is a compact exact symplectic manifold X with a symplectic form ω, where $\omega|_{\partial X} = d\lambda$ for λ a contact form on its boundary. The author associates to a given 4-dimensional Liouville domain (X,ω) a sequence of real numbers $0 = c_0(X,\omega) \le c_1(X,\omega) \le \cdots \le c_k(X,\omega) \le \cdots \le \infty$, called the ECH capacities of (X,ω). The main result of the article under review is that if (X_0,ω_0) and (X_1,ω_1) are two Liouville domains such that (X_0,ω_0) symplectically embeds into (X_1,ω_1), then $c_k(X_0,\omega_0) \le c_k(X_1,\omega_1)$ for every $k \ge 0$ with strict inequality when $c_k(X_0,\omega_0) < \infty$.

The ECH capacities of a Liouville domain (X,ω) are defined to be the ECH spectrum of its boundary (Y,λ). The latter is defined via the action filtration on ECH which measures the length of admissible orbit sets that generate the ECH chain complex. Suppose (Y,λ) is a connected contact 3-manifold with non-vanishing ECH contact invariant, which is represented by the empty orbit set $\emptyset$. Having fixed nonzero $\sigma \in \mathrm{ECH}(Y,\lambda,0)$, the author defines the quantity $c_\sigma(Y,\lambda)$ as the infimum of $L \ge 0$ for which σ is in the image of the map $\mathrm{ECH}^L(Y,\lambda,0) \to \mathrm{ECH}(Y,\lambda,0)$. Then the ECH spectrum of (Y,λ) is the sequence $\{c_k(Y,\lambda)\}_{k\in\mathbb{N}}$, where

$$c_k(Y,\lambda) := \min\{c_\sigma(Y,\lambda) | \sigma \in \mathrm{ECH}(Y,\lambda,0),\ U^k\sigma = [\emptyset]\}.$$

The assumption that Y is connected is redundant. The above definition can be generalized to disconnected contact 3-manifolds as well.

With the preceding understood, the proof of the main result of the article under review exploits what is known about cobordism maps on ECH induced by weakly exact symplectic cobordisms, that is, a cobordism from (Y_+, λ_+) to (Y_-, λ_-) with an exact symplectic form ω such that $\omega|_{Y_\pm} = d\lambda_\pm$. These cobordism maps were defined and studied in ["Proof of the Arnold chord conjecture in three dimensions II", preprint, arXiv:1111.3324] by C. H. Taubes and the author using the isomorphism between ECH and Seiberg–Witten Floer cohomology due to Taubes [Geom. Topol. **14** (2010), no. 5, 2497–2581; MR2746723; Geom. Topol. **14** (2010), no. 5, 2583–2720; MR2746724; Geom. Topol. **14** (2010), no. 5, 2721–2817; MR2746725; Geom. Topol. **14** (2010), no. 5, 2819–2960; MR2746726; Geom. Topol. **14** (2010), no. 5, 2961–3000; MR2746727]. Among other things, these maps preserve the action filtration and intertwine the U-action on ECH. As a result, it follows easily from the definition that if there exists a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-), then $c_k(Y_+, \lambda_+) \geq c_k(Y_-, \lambda_-)$ for each $k \in \mathbb{N}$. That said, the proof of the main result follows at once from the observation that if a Liouville domain (X_0, ω_0) embeds into another Liouville domain (X_1, ω_0), then $X_1 \backslash \mathrm{int}(X_0)$ is a weakly exact symplectic cobordism.

In a significant portion of the article, the author investigates the ECH capacities of various examples including ellipsoids and polydisks, and finds numerical embedding obstructions for these examples. It is worth noting that D. McDuff and F. Schlenk showed that the ECH embedding obstructions of one ellipsoid into another are sharp [Ann. of Math. (2) **175** (2012), no. 3, 1191–1282; MR2912705]. The author also states and discusses a conjecture about determining the symplectic volume of a Liouville domain via its ECH capacities. More precisely, if (X, ω) is a 4-dimensional Liouville domain with $c_k(X, \omega) < \infty$ for all $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} \frac{c_k(X, \omega)^2}{k} = 4\,\mathrm{vol}(X, \omega).$$

The author verifies this conjecture in various cases. As a matter of fact, D. Cristofaro-Gardiner, V. Gripp, and the author recently announced a proof of this conjecture ["The asymptotics of ECH capacities", preprint, arXiv:1210.2167].

This article is very much self-contained and presents an interesting application of ECH to symplectic geometry. Moreover, the fact that it investigates various examples in detail makes it quite instructive as to the use of ECH.

Çağatay Kutluhan

From MathSciNet, January 2018

MR2912705 53D42; 11B39, 53D35

McDuff, Dusa; Schlenk, Felix

The embedding capacity of 4-dimensional symplectic ellipsoids.

Annals of Mathematics. Second Series **175** (2012), *no.* 3, 1191–1282.

Given a real number $a \geq 1$ consider the ellipsoid

$$E(1, a) = \left\{ {x_1}^2 + {x_2}^2 + \frac{{x_3}^2 + {x_4}^2}{a} \leq 1 \right\} \subset \mathbb{R}^4.$$

The paper under review studies the graph of the function $c\,[1,\infty) \to \mathbb{R}$ defined by posing $c(a)$ to be the infimum of all real numbers μ for which there exists a symplectic embedding of $E(1,a)$ into the ball $B(\mu) = \{\, x_1^2 + x_2^2 + x_3^2 + x_4^2 \le \mu \,\}$, with respect to the standard symplectic form $\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$.

An elementary fact about the function c is that $c(a) \ge \sqrt{a}$ for all a, because every symplectic transformation preserves the volume. However, due to Gromov's non-squeezing theorem it is known that preservation of volume is not the only constraint for the symplectic embedding problem. As was later understood, other constraints come from the characteristic flow on the boundary of a domain, and are seen, for example, by the Ekeland-Hofer capacities. Although it seemed reasonable to believe that the volume and the Ekeland-Hofer capacities might give a complete set of obstructions for the symplectic embedding problem of ellipsoids in Euclidean space, D. McDuff discovered in [J. Topol. **2** (2009), no. 1, 1–22; MR2499436] that at least in dimension 4 this is not the case: she proved that there are indeed also other obstructions, related to the theory of embedded contact homology developed by M. L. Hutchings [J. Differential Geom. **88** (2011), no. 2, 231–266; MR2838266].

In the paper under review the authors calculate $c(a)$ for all values of a, and using this calculation they show that the capacities coming from embedded contact homology give sharp obstructions for the problem of symplectically embedding an ellipsoid $E(1,a)$ into a ball, thus confirming in this special case a conjecture of Hofer. Note that the full Hofer conjecture on 4-dimensional ellipsoids has been proved by McDuff in a more recent paper [J. Differential Geom. **88** (2011), no. 3, 519–532; MR2844441], without using the results in the paper under review.

The structure of the graph of the function c is very rich, and turns out to be related to Fibonacci numbers, weight expansions and continued fractions, exceptional curves in blow-ups of $\mathbb{CP}^2$ and to the problem of counting lattice points in right-angled triangles.

The main result of the paper can be described by saying that the graph of c is composed of the following three parts:

- On the interval $[1,\tau^4]$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, the graph of c forms an infinite Fibonacci staircase converging to the point (τ^4,τ^2). More precisely, let g_n, $n \in \mathbb{N}$, denote the odd terms in the sequence of Fibonacci numbers and let $a_n = (\frac{g_{n+1}}{g_n})^2$ and $b_n = \frac{g_{n+2}}{g_n}$. Then on the intervals $[a_n, b_n]$ the graph of c is a line through the origin, with slope $\frac{1}{\sqrt{a_n}}$, while on the intervals $[b_n, a_{n+1}]$ it is the horizontal line $c \equiv \sqrt{a_{n+1}}$ (note that $\frac{b_n}{\sqrt{a_n}} = \sqrt{a_{n+1}}$ so that the function is continuous).

- For $a \ge 8\,\frac{1}{36}$ we have that $c(a) = \sqrt{a}$ (i.e. volume is the only obstruction in this region).

- On the interval $[\tau^4, 8\,\frac{1}{36}]$ we have a transition region: $c(a) = \sqrt{a}$ except on a finite number of short intervals (described explicitly in the paper).

The main ideas behind the proof of this result are the following (see also the review articles by McDuff [Jpn. J. Math. **4** (2009), no. 2, 121–139; MR2576029] and Hutchings [Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8093–8099; MR2806644]):

Using a result of McDuff [op. cit.; MR2499436], the problem of symplectically embedding an ellipsoid into a ball can be reduced to the problem of symplectically embedding a disjoint union of balls into a ball. More precisely, let $a \ge 1$ be rational and let $w(a) = \big(w_1(a), \dots, w_k(a)\big)$ be its weight expansion. Then $E(1,a)$ can be symplectically embedded into $B(\mu)$ if and only if the disjoint union $\bigsqcup_i B(w_i(a))$

can be symplectically embedded into $B(\mu)$. Recall that the weight expansion of a rational number $a = \frac{p}{q}$ (written in lowest terms) can be thought of as describing how to subdivide a rectangle of sides p and q into squares. For example, $w(\frac{25}{9}) = (1, 1, \frac{7}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9})$: we can subdivide a rectangle of sides 25 and 9 into two squares of side 9, one of side 7, three of side 2 and two of side 1. Moreover, the multiplicity of the weights gives the continued fraction expansion of $\frac{p}{q}$. For example,

$$\frac{25}{9} = 2 + \frac{1}{1 + \frac{1}{3+\frac{1}{2}}}.$$

The result that $E(1, a)$ can be symplectically embedded into $B(\mu)$ if and only if the disjoint union $\bigsqcup_i B(w_i(a))$ can be, follows from the fact that $E(1, a)$ decomposes into the union of balls whose sizes are given by the weights of a, since the decomposition of a rectangle into squares given by the weights induces a decomposition of the moment image of the ellipsoid into moment images of balls. On the other hand, the converse uses Taubes-Seiberg-Witten theory and J-holomorphic curves.

Consider now the set $\mathcal{E}_k$ of tuples $(d; m_1, \dots, m_k)$ with $m_i \geq m_{i+1}$ that satisfy the Diophantine equations $d^2 + 1 = \Sigma m_i^2$ and $3d - 1 = \Sigma_i m_i$ and the additional algebraic condition requiring that $(d; m_1, \dots, m_k)$ can be reduced to $(0; 1, \dots, 1)$ by repeated Cremona moves. The authors of the paper under review prove that $\mathcal{E}_k$ describes the set of homology classes $dL - \Sigma_i m_i E_i$ in the k-fold blow-up of $\mathbb{CP}^2$ that can be represented by a symplectically embedded sphere of self-intersection -1 (here L is the class of a line $\mathbb{CP}^1$ and E_i the class of the i-th exceptional divisor). As proved by McDuff and L. Polterovich [Invent. Math. **115** (1994), no. 3, 405–434; MR1262938] the problem of symplectically embedding a disjoint union of balls $\bigsqcup_i^k B(w_i(a))$ into a ball can be reduced to the problem of understanding the symplectic cone of the k-fold blow-up of $\mathbb{CP}^2$ (i.e. the set of cohomology classes represented by a symplectic form), and by work of McDuff [in *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, 85–99, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998; MR1635697], P. Biran [Geom. Funct. Anal. **7** (1997), no. 3, 420–437; MR1466333], T.-J. Li and A.-K. Liu [J. Differential Geom. **58** (2001), no. 2, 331–370; MR1913946] and B. H. Li and T.-J. Li [Asian J. Math. **6** (2002), no. 1, 123–144; MR1902650] the structure of this cone is understood in terms of the set $\mathcal{E}_k$. As a consequence of these results the authors of the paper under review obtain that for each $a \in \mathbb{Q}$

$$c(a) = \max\left(\sqrt{a}, \sup_{(d;m_1,\dots,m_k)\in\mathcal{E}_k} \frac{\sum m_i w_i(a)}{d}\right). \tag{1}$$

Using (1) it is relatively easy to see that $c(a) = \sqrt{a}$ for $a \geq 8\frac{1}{36}$, while the key to describing the function c on $[1, \tau^4]$ as a Fibonacci staircase is given by the surprising discovery that there exist classes in $\mathcal{E}_k$ which are given by tuples $(d; m_1, \dots, m_k)$ constructed from weight expansions of ratios of odd terms in the sequence of Fibonacci numbers. On the other hand, in order to understand the function c on the transition region $[\tau^4, 8\frac{1}{36}]$ a more delicate analysis is needed, involving among other things a study of the elements of the sets $\mathcal{E}_k$, of the corresponding functions $\sum m_i w_i/d$, of the influence of a ghost staircase made from the even terms of the

Fibonacci sequence, and the derivation of surprising identities satisfied by weight expansions.

Sheila Sandon
From MathSciNet, January 2018

MR3069365 53D35; 57R17

Buse, O.; Hind, R.

Ellipsoid embeddings and symplectic packing stability.

Compositio Mathematica **149** (2013), *no.* 5, 889–902.

The paper under review is a sequel to [O. Buşe and R. Hind, Geom. Topol. **15** (2011), no. 4, 2091–2110; MR2860988], which proved stability results for symplectic ellipsoid embeddings in any dimension. This is fairly remarkable, given the lack of techniques dealing with higher-dimensional packing problems, compared to the situation in dimension 4. In particular, in dimension 4, the stability of symplectic manifolds (M,ω) with $[\omega]\in H^2(M,\mathbb{Q})$ was established a long time ago, by P. Biran [Invent. Math. **136** (1999), no. 1, 123–155; MR1681101].

The main result of the paper under review is stated as follows:

Theorem. There exists a constant $S(b_1,\dots,b_n)$ such that if $a_n/a_1 > S$ and $a_1\cdots a_n \le b_1\cdots b_n$, there exists a symplectic embedding

$$E(a_1,\dots,a_n)\longrightarrow E(b_1,\dots,b_n).$$

Intuitively, this means the only obstruction of embedding into a target ellipsoid is the volume obstruction when the source ellipsoid is thin enough.

A main technical ingredient of the paper is Theorem 1.4, which is a rather handy 4-dimensional result. Theorem 1.4 was proved using a general necessary and sufficient condition of ellipsoid embedding in dimension 4 due to M. L. Hutchings [J. Differential Geom. **88** (2011), no. 2, 231–266; MR2838266] and D. McDuff [J. Differential Geom. **88** (2011), no. 3, 519–532; MR2844441], which is in general not easy to verify.

Another very useful technical ingredient is a refinement of the suspension result from [O. Buşe and R. Hind, op. cit.], i.e. Proposition 3.4 of the paper. This says if $E(a,b)$ embeds into $E(c,d)$, then $E(a,b,a_3,\dots,a_n)$ embeds into $E(c,d,a_3,\dots,a_n)$. The authors combine the above two results to reduce the main argument to inductive rearrangements of two consecutive radii of the ellipsoid.

Given the stability result on ellipsoid embeddings into ellipsoids, it follows that when $[\omega]\in H^2(M,\mathbb{Q})$, a similar stability result holds for the closed symplectic manifold (M,ω). This essentially follows from the generalized Biran decomposition in higher dimensions (this implies that there exists a full ellipsoid embedding into such closed symplectic manifolds), proved by E. Opshtein [Compos. Math. **143** (2007), no. 6, 1558–1575; MR2371382; J. Symplectic Geom. **11** (2013), no. 1, 109–133; MR3022923] and also presented in the current paper (Theorem 4.1).

The paper also computes examples of stability numbers. For example, it is a nice result that $N_{\mathrm{stab}}(\mathbb{CP}^3)$ is shown to lie in a very narrow range between 8 and 21. Explicitly, this means when $n\ge 21$, packing n equal balls has no obstructions other than volume; while one cannot fully embed 7 balls into $\mathbb{CP}^3$ (the latter part is a classical result due to M. Gromov [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718]).

Weiwei Wu
From MathSciNet, January 2018

MR3286479 53D05; 53D10, 53D12

Eliashberg, Yakov

Recent advances in symplectic flexibility.

Bulletin of the American Mathematical Society. New Series **52** (2015), *no.* 1, 1–26.

In this expository article the author reviews the most prominent flexibility results in symplectic and contact topology, starting from M. Gromov's work on the h-principle in the 60's and 70's to E. Murphy's recent discovery of a flexible class of Legendrian submanifolds and the author's joint work with M. S. Borman and Murphy on the classification of overtwisted contact structures in all dimensions.

In the author's words, "flexible and rigid problems and the development of each side toward the other shaped and continue to shape the subject of symplectic topology from its inception". On the flexibility side, the basic starting points are the classical theorems of Darboux, Moser and Gray which state local flexibility and stability of symplectic and contact structures. Moreover, in the 60's and 70's Gromov's work on the h-principle [*Partial differential relations*, Ergeb. Math. Grenzgeb. (3), 9, Springer, Berlin, 1986; MR0864505] showed that many symplectic and contact problems are governed by flexibility. Gromov also proved that either the group of symplectomorphisms is $\mathcal{C}^0$-closed in the diffeomorphism group or its $\mathcal{C}^0$-closure coincides with the group of volume-preserving diffeomorphisms. At the same time Arnold stated his famous conjecture on fixed points of Hamiltonian symplectomorphisms (a statement that was known to be false in general for volume-preserving diffeomorphisms in dimension bigger than two). The Arnold conjecture was first proved in some special cases in the 80's, and at the same time also other phenomena were discovered that solved Gromov's alternative in favor of rigidity (in particular, Gromov's non-squeezing theorem). In the author's words, after the introduction of holomorphic curves by Gromov [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718], "the rigid side of symplectic topology began unravelling with an exponentially increasing speed" and "rigid methods dominated the development of the subject during the last three decades". On the other hand, "flexible milestones after the resolution of Gromov's alternative" are, according to the author, the classification by the author of overtwisted contact structures in dimension three [Invent. Math. **98** (1989), no. 3, 623–637; MR1022310], S. K. Donaldson's almost holomorphic sections method for constructing codimension two symplectic submanifolds in dimensions greater than four [J. Differential Geom. **44** (1996), no. 4, 666–705; MR1438190], the work by the author on existence of Stein structures [Internat. J. Math. **1** (1990), no. 1, 29–46; MR1044658] and L. Guth's flexibility result on symplectic embeddings of polydisks [Invent. Math. **172** (2008), no. 3, 477–489; MR2393077]. After briefly reviewing these works, the author discusses in more detail some recent breakthroughs on symplectic flexibility, in particular the results originated by Murphy's thesis and the classification of overtwisted contact structures in all dimensions by Borman, Murphy and the author.

In ["Loose Legendrian embeddings in high dimensional contact manifolds", preprint, arXiv:1201.2245] Murphy discovered, on all contact manifolds of dimension bigger than three, a class of Legendrian embeddings (which are called *loose*) that are flexible, in the sense that they satisfy a certain h-principle. This recent discovery already led to many important applications. In particular, loose Legendrians are at the base of the work of the author and Murphy [Geom. Funct. Anal. **23** (2013), no. 5, 1483–1514; MR3102911] on Lagrangian caps, which also led to unexpected constructions by T. Ekholm, Murphy, I. Smith and the author [Geom. Funct. Anal. **23** (2013), no. 6, 1772–1803; MR3132903] of Lagrangian immersions with minimal number of self-intersection points. Using loose Legendrians, K. Cieliebak and the author [*From Stein to Weinstein and back*, Amer. Math. Soc. Colloq. Publ., 59, Amer. Math. Soc., Providence, RI, 2012; MR3012475] defined a class of *flexible Weinstein manifolds*, a notion which in turn led to applications to the topology of polynomially and rationally convex domains [K. Cieliebak and Y. M. Eliashberg, Invent. Math. **199** (2015), no. 1, 215–238; MR3294960] and was also a major ingredient in S. Courte's negative answer to the question of whether contact manifolds with exact symplectomorphic symplectization are necessarily contactomorphic [Geom. Topol. **18** (2014), no. 1, 1–15; MR3158770]. Finally, in the recent preprint ["Existence and classification of overtwisted contact structures in all dimensions", preprint, arXiv:1404.6157] Borman, Murphy and the author generalized the author's work in dimension three [op. cit.; MR1022310] to prove that any almost contact structure on a closed manifold (of any dimension) is homotopic to a contact structure. Moreover they extended the definition of overtwisted contact structures to all dimensions and proved that on any closed manifold any almost contact structure is homotopic to an overtwisted contact structure, which is unique up to isotopy.

The paper under review is an expanded version of [Y. M. Eliashberg, in *The influence of Solomon Lefschetz in geometry and topology*, 3–18, Contemp. Math., 621, Amer. Math. Soc., Providence, RI, 2014; MR3289318]. The most important additions are a review of the classification of overtwisted contact structures in all dimensions that appeared after the publication of the first paper, and a discussion in the last section of further directions of research.

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