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COMMENTARY ON "ERGODIC THEORY OF AMENABLE GROUP ACTIONS": OLD AND NEW

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Abstract. Immediately following the commentary below, this previously published article is reprinted in its entirety: D. Ornstein and B. Weiss, "Ergodic theory of amenable group actions, I. The Rohlin Lemma", *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), no. 1, 161–164.

Up to a small error, an ergodic measure preserving action of a group on a probability space can be approximated by a periodic action. For $\mathbb{Z}$-actions this was proven by Kakutani [14] and Rohlin [23], and this statement is generally referred to as the Rohlin Lemma. More precisely, assume that $(X, \mathcal{X}, \mu, T)$ is a measure preserving system, meaning that $(X, \mathcal{X}, \mu)$ is a probability space and $T\colon X \to X$ is a measurable, measure preserving, and invertible transformation. Thus the transformation T acting on the space $(X, \mathcal{X}, \mu)$ gives rise to a $\mathbb{Z}$-action. Further assume that this system is ergodic, meaning that the only measurable sets $A \in \mathcal{X}$ such that $TA = A$ either have measure 0 or have measure 1. The Rohlin Lemma states that for an ergodic measure preserving system $(X, \mathcal{X}, \mu, T)$, any $\varepsilon > 0$, and $N \in \mathbb{N}$, there exists a set $F \in \mathcal{X}$ such that $T, TF, T^2F, \ldots, T^{N-1}F$ are pairwise disjoint and such that

$$\mu\left(\bigcup_{i=0}^{N-1} T^n F\right) > 1 - \varepsilon.$$

Thus by changing the transformation on an arbitrarily small portion of the space, any two $\mathbb{Z}$-actions are indistinguishable. Halmos [10] showed that aperiodicity suffices to prove the result. The Rohlin Lemma is a basic tool for understanding measurable actions of $\mathbb{Z}$ on a measure space, leading to classification results and providing a basic mechanism for numerous constructions in ergodic theory.

The standard proofs of the Rohlin Lemma rely on the ordering of $\mathbb{Z}$, and new methods were developed to extend the result, generalizing the group acting on the measure preserving space (see, for example [5], [15], [8]). There were significant hurdles in extending the result to more general group actions, and these were overcome via the introduction by Ornstein and Weiss [21] (the article is reprinted following this commentary) of the tiling and quasi-tiling machinery. The main result in their short article extends the Rohlin Lemma to amenable group actions, and this continues to play a fundamental role in the development of ergodic theory beyond the action of a single transformation.

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A countable amenable group has a sequence of approximately invariant sets that exhaust the space; such sets are the analogue of the intervals $[-N, N]$ in $\mathbb{Z}$. As many ergodic arguments involve averaging, these sets allow one to average over the group and provide the natural setting in which to extend the ergodic theory of $\mathbb{Z}$-actions to more general groups. A first question is whether amenable actions exhibit fundamentally different behavior from that of $\mathbb{Z}$-actions, and Ornstein and Weiss showed that up to a small error, all such actions, again, are all indistinguishable. As an immediate corollary, one has a strengthening of Dye's Theorem [7] to amenable groups, showing that any two amenable actions are equivalent up to an isomorphism that carries orbits into orbits. Given this extension of the Rohlin Lemma to amenable groups, it is natural to use it as a tool to extend other results known for $\mathbb{Z}$-actions to the more general setting of amenable actions.

In this vein, Ornstein's entropy and isomorphism theory was extended from $\mathbb{Z}$-actions to $\mathbb{Z}^d$-actions in [12] and [13], and one of the first applications of the more general Rohlin Lemma was in the deep work of Ornstein and Weiss [22], extending the theory to the setting of amenable group actions. This *tour de force* includes full proofs of the tiling results announced in [21], uses of these tiling results to prove versions of the Rohlin Lemma for discrete and for continuous groups, extensions of Ornstein's isomorphism theory to amenable actions, and the broad development of entropy theory for amenable group actions. Since then other proofs have been given for the Ornstein–Weiss generalization of the Rohlin Lemma that is at the heart of this work, including proofs in [20] and [9].

The Rohlin Lemma and generalized versions continue to be developed and extended, and they continue to be a major ingredient in modern developments. A version of the quasi-tiling technique plays a role in Lindenstrauss's results [19] on pointwise ergodic theorems for amenable group actions. Hjorth [11] proved that Ornstein and Weiss's result in [21] characterizes amenability, providing a converse to Dye's Theorem. Recently, Bowen [1] showed that the extension of the Ornstein–Weiss isomorphism theory for Bernoulli shifts passes beyond amenable groups, continuing the classification results for free groups, among others (see for example [2], [3], [16], [17], [18]). Further strengthenings of the Rohlin Lemma have been proven, including replacing the approximate quasi-tilings with exact tilings in [6], [4], and these results continue to yield new insights into the classification and structure of general group actions.

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