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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ANDRÉ WEIL

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Dwork, Bernard

On the rationality of the zeta function of an algebraic variety.

American Journal of Mathematics **82** (1960), 631–648.

Das Hauptergebnis ist der folgende Satz: Die Zetafunktion $\zeta(V,t)$ einer algebraischen Mannigfaltigkeit V über einem endlichen Körper k ist eine rationale Funktion von t (Vermutung von Weil). $\zeta(V,t)$ ist durch

$$\zeta(V,t)=\exp[\sum_{i=1}^{\infty}N_i t^i/i] \tag{1}$$

definiert, wo N_i die Anzahl der Punkte von V bedeutet, die Koordinaten in der Erweiterung i-ten Grades k_i von k haben. V mag eine affine, projektive oder allgemeiner eine abstrakte Mannigfaltigkeit im Sinne von Weil sein, V darf auch reduzibel und singularitätenbehaftet sein. Durch naheliegende kombinatorische Überlegungen wird der Beweis auf den Fall zurückgeführt, daß V die Differenz einer Hyperfläche $f(x_1,\cdots,x_n)=0$ im n-dimensionalen affinen Raume über k und der reduziblen Hyperfläche $\prod_{i=1}^n x_i=0$ ist, so daß also N_i die Anzahl der Lösungen $x_1,\cdots,x_n$ von $f(x_1,\cdots,x_n)=0$ mit $x\in {k_i}^{\times}$ (Multiplikativgruppe von k_i) ist. Mit irgend einem nichttrivialen Charakter Θ der Additivgruppe von k_i gilt

$$q^i N_i=(q^i-1)^n+\sum_{x_\nu\in k_i;\nu=0,1,\cdots,n}\Theta(x_0 f(x_1,\cdots,x_n)), \tag{2}$$

wo $q=p^a$ die Elementezahl von k sei, p die Charakteristik von k. Dies wird in die p-adische Approximationsformel

$$\begin{aligned}(3)\quad & q^i N_i=(q^i-1)^n+\\ &\sum_{\xi_\nu\in T_i;\nu=0,1,\cdots,n}\prod_{j=0}^{i-1}F_r({\xi_0}^{q^j},\cdots,{\xi_n}^{q^j}) \bmod p^r,\ r=1,2,3,\cdots,\end{aligned}$$

übergeführt, $F_r(X_0,\cdots,X_n)$ ist ein Polynom mit ganzen Koeffizienten in dem Körper Ω, der vollständigen und algebraisch abgeschlossenen Hülle des Körpers $\mathbf{Q}_p$ der rationalen p-adischen Zahlen und T_i ist die Gruppe der (q^i-1)-ten Einheitswurzeln in Ω. Diese Umformung verläuft folgendermaßen: $\xi\to x$ sei der Isomorphismus von ${k_i}^{\times}$ auf T_i, der ξ seine Restklasse x nach dem maximalen Ideal zuordnet, order umgekehrt: $\xi=\xi(x)$ ist der multiplikative (Teichmüller-) Repräsentant von x in Ω. Ist ζ eine primitive p-te Einheitswurzel und ist L_i die unverzweigte Erweiterung i-ten Grades von $\mathbf{Q}_p$, so ist $\Theta(x)=\zeta^{\mathrm{Spur}_{L_i/\mathbf{Q}_p}(\xi(x))}$ ein nichtrivialer Charakter von ${k_i}^{+}$. Für diesen wird eine Aufspaltung $\Theta(x)=\prod_{j=0}^{i-1}\theta(\xi(x)^{p^j})$ hergeleitet, in der $\theta(t)=\sum_{m=0}^{\infty}\beta_m t^m$ eine Potenzreihe bedeutet, deren Koeffizienten β_m (aus Ω) die Abschätzung $|\beta_m|\le|p|^{m/(p-1)}$ erfüllen ($|\ |$ bedeutet den Betrag im bewerteten Körper Ω). Wird dies noch dem Körper k durch

Einführung von $\Lambda(t) = \prod_{j=0}^{a-1} \theta(t^{p^j}) = \sum_{m=0}^{\infty} \lambda_m t^m$ angepaßt—es gilt wieder (4) $|\lambda_m| \leq |p|^{m/(p-1)}$—so folgt aus (2)

$$q^i N_i = (q^i - 1)^n + \sum_{\xi_\mu \in T_i} \prod_{\nu=1}^{\rho} \prod_{j=0}^{i-1} \Lambda(A_\nu M_\nu {\xi_\mu}^{q^j})$$

wenn $X_0 F(X_1, \cdots, X_n) = \sum_{\nu=1}^{\rho} A_\nu M_\nu$ mit $A_\nu \in k$, M_ν Potenzprodukt der X_μ ist. Da die $\Lambda(A_\nu {M_\nu}^{q^j})$ keine Polynome sind, ist es nötig $\sum_{m=0}^{r(q-1)} \lambda_m t^m = \Lambda_r(t)$ gesetzt, für $= 1, 2, \cdots,$ Polynome (5) $F_r(X) = \sum_r \Lambda_r(A_\nu M_\nu)$ einzuführen, mit diesen gilt dann (beachte (4)) die Formel (3). Für die Summe rechts in (3) hat man nun nach des Verfassers Arbeit [siehe #3913] einen Ausdruck als Spur einer linearen Transformation des Polynomrings $\Omega[X]$ in sich

$$\sum_{\xi_\nu \in T_i; \nu=0,1,\cdots,n} \prod_{j=0}^{i-1} F_r({\xi_0}^{q^j}, \cdots, {\xi_n}^{q^j}) = (q^i - 1)^{n+1} \text{ Spur } (\psi \circ F_r)^i.$$

Ähnlich wie [loc. cit.] schließt man jetzt auf die Existenz des Grenzwertes (6) $\lim_{r\to\infty} \det(I - t\psi \circ F_r) = \Delta(t)$, dabei ist die Konvergenz—im Potenzreihenring $\Omega\{t\}$—koeffizientenweise p-adisch gemeint; weiter, daß (7) $\zeta(V, qt) = (1 - t)^{-(-\delta)^n} \Delta(t)^{-(-\delta)^{n+1}}$ gilt, wo δ den loc. cit. erklärten (in der eben erwähnten Topologie) topologischen Automorphismus der multiplikativen Gruppe $1 + t\Omega\{t\}$ bedeutet. Man kann nun die Koeffizienten von $\det(I - t\psi \circ F_r)$ aus der Definition (5) von F_r berechnen und dann, dank der Abschätzung (4), zeigen, daß für die Koeffizienten von $\det(I - t\psi \circ F_r) = \sum_{m=0}^{\infty} \gamma_{r,m} t^m$ eine Abschätzung $|\gamma_{r,m}|^{1/m} \leq \varepsilon_m$; $r = 1, 2 \cdots,$ mit $\lim_m \varepsilon_m = 0$ gilt. Für $\Delta(t) = \sum_{m=0}^{\infty} \gamma_m t^m$ folgt $|\gamma_m|^{1/m} \leq \varepsilon_m$, und also ist $\Delta(t)$ (in Ω) beständig konvergent; (7) ergibt, daß $\zeta(V, t)$ p-adisch meromorph (Quotient von zwei beständig konvergenten Potenzreihen in Ω ist.

Bemerkt man nun noch, daß $\zeta(V, t)$ zufolge der Definition (1) eine Potenzreihe mit ganzen rationalen Koeffizienten ist, so folgt die Rationalität von $\zeta(V, t)$ mit Hilfe des auch an sich interessanten Kriteriums: Eine Potenzreihe $F(t) = \sum_{i=0}^{\infty} A_i t^i$ mit Koeffizienten A_i aus einem endlichen algebraischen Zahlkörper L ist genau dann rational, wenn die Menge der Primstellen $\mathfrak{p}$ von L so in eine endliche Menge S und ihr Komplement S' eingeteilt werden kann, daß (i) $|A_i|_\mathfrak{p} \leq 1$, $i = 0, 1, 2, \cdots,$ gilt für $\mathfrak{p} \in S'$; (ii) $F(t)$ als Funktion einer $\mathfrak{p}$-adischen Variabeln $\tau_\mathfrak{p}$ in $\Omega_\mathfrak{p}$ (der algebraisch abgeschlossenen und $\mathfrak{p}$-adisch vollständigen Hülle von L) in einem Kreise $|\tau_\mathfrak{p}|_{\mathfrak{p}\leq} R_\mathfrak{p}$ meromorph (d.h. Quotient zweier dort konvergenter Potenzreihen) ist, und es gilt $\prod_{\mathfrak{p}\in S} R_{\mathfrak{p}>1}$. Die $\mathfrak{p}$-Beträge sind dabei so normiert zu denken, daß für $A \neq 0$ aus L die Produktformel $\prod_\mathfrak{p} |A|_\mathfrak{p} = 1$ gilt. Der Beweis beruht auf dem klassischen Kriterium für die Rationalität einer Potenzreihe $\sum_{i=0}^{\infty} A_i t^i$ mit Koeffizienten aus einem Körper (von É. Borel): Genau dann ist $F(t)$ rational, wenn es ein $m = 1, 2, \cdots$ und ein $i_0 = 1, 2, \cdots,$ gibt, so daß die Hankelschen Determinanten $N_{i,m} = \det{(A_{i+j+1})}_{j,1=0,\cdots,m}$ null sind für $i \geq i_0$. Es wird übrigens nur der einfachste Fall $L = \mathbf{Q}$ und $S = \{p, p_\infty\}$ dieses Kriteriums gebraucht. Zum Schluß wird noch auf die loc. cit. nur unter der Annahme der Rationalität von $\zeta(V, t)$ bewiesenen Aussagen hingewiesen.

Die Aufspaltung des additiven Charakters Θ mittels der gut konvergenten Potenzreihe θ, die ein wesentliches Hilfsmittel des Beweises bildet, ist, wie der Verfasser bemerkt, keinesweges die einzig mögliche und es wird angedeutet, wie man sie auch auf anderem Wege gewinnen kann; ferner wird bemerkt, daß die genannte Aufspaltung auch für die Theorie der Gaußschen Summen nützlich ist, es wird eine Kongruenz von Stickelberger [Math. Ann. **37** (1890), 321–367] mit ihrer Hilfe hergeleitet.

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Deligne, Pierre

La conjecture de Weil. I. (French)

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This is without question the most important paper in algebraic geometry to have appeared in the last ten years (since H. Hironaka's proof of resolution of singularities in characteristic zero [Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326; MR0199184]). Deligne has proved the Riemann hypothesis for varieties over finite fields! The superficial reader may be dismayed that the proof uses the full strength of the Artin-Grothendieck étale cohomology theory as it is set down in the 2544 pages of SGA4 and SGA7 [*Théorie des topos et cohomologie étale des schémas, Tome 1: Théorie des topos* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 269, Springer, Berlin, 1972; *Théorie des topos et cohomologie étale des schémas, Tome 2* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 270, Springer, Berlin, 1972; *Théorie des topos et cohomologie étale des schémas, Tome 3* (Sém. de Géométrie Algébrique du Bois-Marie 1963–1964, SGA4), Lecture Notes in Math., Vol. 305, Springer, Berlin, 1973; *Groupes de monodromie en géométrie algébrique, I* (Sém. de Géométrie Algébrique du Bois-Marie 1967–1969, SGA7I), Lecture Notes in Math., Vol. 288, Springer, Berlin, 1972; *Groupes de monodromie en géométrie algebrique, II* (Sém. de Géométrie Algébrique du Bois-Marie 1967–1969, SGA7II), Lecture Notes in Math., Vol. 340, Springer, Berlin, 1973] and the various loose exposés (available in preprint form only) of the still unpublished SGA5. In fact the author does an excellent job of isolating and stating clearly precisely which results he is using, thus providing the reader with a valuable skeleton key to those weighty tomes. Because the actual proof is quite readable, and has already been summarized by J.-P. Serre ("Valeurs propres des endomorphismes de Frobenius (d'après P. Deligne)", to appear in Séminaire Bourbaki, 1973/1974, 26$^{\text{ème}}$ année), I will only discuss the sources of some of the ideas that go into the proof.

Recall that for any variety X over a finite field $\mathbf{F}_q$, its zeta function $Z(X/\mathbf{F}_q, T)$ is defined as the formal power series $\exp(\sum_{n\geq 1} N_n T^n/n)$, where N_n is the number of points of X with coordinates in the field $\mathbf{F}_{q^n}$. Thus the zeta-function of X provides a sort of Diophantine summary of X.

In 1949, A. Weil [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393] made his famous conjectures about the zeta-function of a projective, non-singular n-dimensional variety X over $\mathbf{F}_q$ (generalizing what he himself had proved for X a curve, an abelian variety or a Fermat hypersurface).

(1) $Z(X/\mathbf{F}_q, T)$ is a rational function of T.

(2) Moreover,

$$Z(X/\mathbf{F}_q, T) = P_1(T)P_3(T)\cdots P_{2n-1}(T)/P_0(T)P_2(T)\cdots P_{2n}(T),$$

where $P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}T)$, $|\alpha_{ij}| = q^{i/2}$, the last equality being the "Riemann hypothesis" for varieties over finite fields.

(3) Under $\alpha \mapsto q^n/\alpha$, the $\alpha_{i,j}$ are carried bijectively to the $\alpha_{2n-i,j}$. This is a functional equation for $T \mapsto 1/q^nT$.

(4) In case X is the "reduction modulo p" of a nonsingular projective variety $\mathbf{X}$ in characteristic zero, then b_i is the ith topological Betti number of $\mathbf{X}$ as a complex manifold.

The moral is that the topology of the complex points of $\mathbf{X}$, expressed through the classical cohomology groups $H^i(\mathbf{X}, \mathbf{C})$, determines the form of the zeta-function of X, i.e., determines the Diophantine shape of X. Weil gave a heuristic argument for this, as follows [Proceedings of the International Congress of Mathematicians (Amsterdam, 1954), Vol. III, pp. 550–558, Noordhoff, Groningen, 1956; MR0092196]. Among all elements of the algebraic closure of $\mathbf{F}_p$, the elements of $\mathbf{F}_q$ are singled out as the fixed points of the Frobenius morphism $x \mapsto x^q$. More generally, if $\mathbf{x} = (\cdots, x_i, \cdots)$ is a solution of some equations which are defined over $\mathbf{F}_q$, then $F(\mathbf{x}) \overset{\text{dfn}}{\to} = (,\cdots, {x_i}^q, \cdots)$ will also be a solution of the same equations, and the point $\mathbf{x}$ will have its coordinates in $\mathbf{F}_q$ precisely when $F(\mathbf{x}) = \mathbf{x}$. Thus F is an endomorphism of our variety X over $\mathbf{F}_q$, and $N_n = \#\mathrm{Fix}(F^n)$; thus $\mathrm{Z}(X/\mathbf{F}_q, T) = \exp(\sum(T^n/n)\#\mathrm{Fix}(F^n))$.

Suppose that we consider instead a compact complex manifold $\mathbf{X}$, and an endomorphism $\mathbf{F}$ of $\mathbf{X}$ with reasonable fixed points. Then the Lefschetz fixed point formula would give us $\#\mathrm{Fix}(\mathbf{F}^n) = \sum(-1)^i \mathrm{trace}(\mathbf{F}^n | H^i(\mathbf{X}, \mathbf{C}))$, which is formally equivalent to the identity

$$\exp(\sum_{n\geq 1}(T^n/n)\#\mathrm{Fix}(\mathbf{F}^n)) = \prod_{i=0}^{2n}\det(1 - T\mathbf{F}|\, H^i(\mathbf{X}, \mathbf{C}))^{(-1)^{i+1}}.$$

The search for a "cohomology theory for varieties over finite fields" which could justify this heuristic argument has been responsible, directly and indirectly, for much of the tremendous progress made in algebraic geometry during the past twenty-five years. Weil's proofs of the Riemann hypothesis for curves over finite fields had already necessitated his *Foundations of algebraic geometry* [Amer. Math. Soc., New York, 1946; MR0023093; revised edition, Providence, R.I., 1962; MR0144898]. Around the same time, O. Zariski had also begun emphasizing the need for an abstract algebraic geometry. His disenchantment with the lack of rigor in the Italian school had come after writing his famous monograph *Algebraic surfaces* [Springer, Berlin, 1935; Zbl **10**, 377] which gave the "state of the art" as of 1934. The possibility of transposing to abstract algebraic varieties with their "Zariski topology" the far-reaching topological and sheaf-theoretic methods that had been developed by Picard, Lefchetz, Hodge, Kodaira, Leray, Cartan,... in dealing with complex varities was implicit in Weil's lecture notes "Fibre spaces in algebraic geometry" [mimeographed lecture notes, Math. Dept., Univ. of Chicago, Chicago, Ill., 1952 (1955)]. This transposition was carried out by Serre in his famous article FAC [Ann. of Math. (2) **61** (1955), 197–278; MR0068874]. From the point of view of the Weil conjectures, however, this theory was still inadequate, for when applied to varieties in characteristic p it gave cohomology groups that were vector spaces in characteristic p, so could only give "mod p" trace formulas, i.e., could only give "mod p" congruences for numbers of rational points.

(I) l-**adic cohomology**: After some false starts (e.g., Serre's Witt vector cohomology [International Symposium on Algebraic Topology, pp. 24–53, Univ. Nac. Autónom. de México, Mexico City, 1958; MR0098097; Amer. J. Math. **80** (1958), 715–739; MR0098100]) and B. M. Dwork's "unscheduled" (because apparently non-cohomological) proof [ibid. **82** (1960), 631–648; MR0140494] of the rationality conjecture (1) for any-variety over $\mathbf{F}_q$, M. Artin and A. Grothendieck developed a "good" cohomology theory based on the notion of étale covering space, and generalizing Weil's l-adic matrices [see the third, fourth and fifth references to SGA4 above]. In fact, they developed a whole slew of theories, one for each prime number $l \neq p$, whose coefficient field was the field $\mathbf{Q}_l$ of l-adic numbers. Each theory gave a factorization of the zeta-function $\mathrm{Z}(T) = \prod_{i=0}^{2n} P_{i,l}(T)^{(-1)^{i+1}}$ into an alternating product of $\mathbf{Q}_l$-adic polynomials, satisfying conjecture (3). In the case when $\mathbf{X}$ could be lifted to $\mathbf{X}$ in characteristic zero, they proved that $P_{i,l}$ was a polynomial of degree $b_i(\mathbf{X})$. However, they did not prove that the $P_{i,l}$ in fact had coefficients in $\mathbf{Q}$, nor a fortiori that the $P_{i,l}$ were independent of l. This meant that in the factorization of an individual $P_{i,l}$, $P_{i,l}(T) = \prod_{j=1}^{b_i}(1 - \alpha_{i,j,l}T)$, the roots $\alpha_{i,j,l}$ were only algebraic over $\mathbf{Q}_l$, but possibly not algebraic over $\mathbf{Q}$, and so they might not even have archimedean absolute values. (Of course, by a theorem of Fatou, the actual reciprocal zeros and poles of the rational function $\mathrm{Z}(T)$ are algebraic integers; the problem is that there may be cancellation between the various $P_{i,l}$ in the l-adic factorization of the zeta-functions.)

So the question became one of how to introduce archimedean considerations into the l-adic theory. Even before the l-adic theory had been developed, Serre [Ann. of Math. (2) **7** (1960), 392–394; MR0112163; correction, MR **22**, p. 2545], following a suggestion of Weil [see the tenth reference above, p. 556], had formulated and proved a Kählerian analogue of the Weil conjectures, making essential use of the Hodge index theorem. In part inspired by this, in part by his own earlier (1958) realization that the Castelnuovo inequality used by Weil was a consequence of the Hodge index theorem on a surface, Grothendieck in the early sixties formulated some very difficult positivity and existence conjectures about algebraic cycles, the so-called "standard conjectures" [cf. S. Kleiman, *Dix exposés sur la cohomologie des schémas*, pp. 359–386, North-Holland, Amsterdam, 1968; MR0292838], whose truth would imply the independence of l and the Riemann hypothesis.

Much to everyone's surprise, the author managed to avoid these conjectures altogether, except to deduce one of them from the Weil conjectures, the "hard" Lefschetz theorem giving the existence of the "primitive decomposition" of the cohomology of a projective non-singular variety, a result previously known only over $\mathbf{C}$, and there by Hodge's theory of harmonic integrals. The rest of the "standard conjectures" remain open. In fact, the generally accepted dogma that the Riemann hypothesis could not be proved before these conjectures had been proved [cf., J. Dieudonné, *Cours de géometrie algébrique, Vol. I: Aperçu historique sur le development de la géométrie algébrique*, especially p. 224, Presses Univ. France, Paris, 1974; *Vol. II: Précis de géométrie algébrique élémentaire*, 1974] probably had the effect of delaying for a few years the proof of the Riemann hypothesis.

(II) **The new ingredients**: So what was it that finally allowed the Riemann hypothesis for varieties over finite fields to be proved? There were two principal ingredients. (1) Monodromy of Lefschetz pencils: In the great work of S. Lefschetz [*L'analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, 1924; reprinting, 1950; MR0033557] on the topology of algebraic varieties, he introduced the technique of systematically "fibering" a projective variety by its hyperplane sections, and then expressing the cohomology of that variety in terms of the cohomology of those fibers. The general Lefschetz theory was successfully transposed into l-adic cohomology, but it didn't really bear Diophantine fruit until D. A. Každan and G. A. Margulis proved that the "monodromy group" of a Lefschetz pencil of odd fibre dimension was as "large as possible". The author realized that if the same result were true in even fibre dimension as well, then it would be possible to inductively prove the independence of l and the rationality of the $P_{i,l}$ of X, by recovering them as generalized "greatest common divisors" of the $P_{i,l}$ of the hyperplane sections. But the Každan-Margulis proof was Lie-algebra theoretic in nature, via the logarithms of the various Picard-Lefschetz transformations in the monodromy group. The restriction to odd fibre dimension was necessary because in that case the Picard-Lefschetz transformations were unipotent, thus had interesting logarithms, while in even fibre dimension they were of finite order. Soon thereafter, N. A'Campo [Invent. Math. **20** (1973), 147–169; MR0338436], found a counterexample to a conjecture of Brieskorn that the local monodromy of isolated singularities should always be of finite order. Turning sorrow to joy, Deligne realized that A'Campo's example could be used to construct (non-Lefschetz) pencils which would have unipotent local monodromy. These he used to make the Každan-Margulis proof work in even fibre-dimension as well, and so to establish the "independence of l" and rationality of the $P_{i,l}$ [cf. J.-L. Verdier, Séminaire Bourbaki, 25$^{\text{ème}}$ année (1972/1973), Exp. No. 423, pp. 98–115. Lecture Notes in Math., Vol. 383, Springer, Berlin, 1974].

With this result, the importance of monodromy considerations for Diophantine questions was firmly established. (2) Modular forms, Rankin's method, and the cohomological theory of L-series: In the years after the Weil conjectures were first formulated, experts in the theory of modular forms began to suspect a strong relation between the Weil conjectures and the Ramanujan conjecture on the order of magnitude of $\tau(n)$. Recall that the $\tau(n)$ are the q-expansion coefficients of the unique cusp form Δ of weight twelve on $\mathrm{SL}_2(\mathbf{Z})$: $\Delta(q) = q(\prod_{n\geq 1}(1-q^n))^{24} = \sum \tau(n)\cdot q^n$. As an arithmetic function, $\tau(n)$ occurs essentially as the error term in the formula for the number of representations of n as a sum of 24 squares. The Ramanujan conjecture is that $|\tau(n)| \leq n^{11/2}d(n)$, $d(n) = \#$ (divisors of n). According to Hecke theory (which had been "prediscovered" by Mordell for Δ), the Dirichlet series corresponding to Δ admits an Euler product: $\sum_{n\geq 1}\tau(n)\cdot n^{-s} = \prod_p(1/1-\tau(p)\cdot p^{-s}+p^{11-2s})$.

The truth of the Ramanujan conjecture for all $\tau(n)$ is then a formal consequence of its truth for all $\tau(p)$ with p prime: $|\tau(p)| \leq 2p^{11/2}$. This last inequality may be interpreted as follows. Consider the polynomial $1-\tau(p)T+p^{11}T^2$ and factor it: $1-\tau(p)T+p^{11}T^2 = (1-\alpha(p)T)(1-\beta(p)T)$. Then the Ramanujan conjecture for $\tau(p)$ is equivalent to the equality $|\alpha(p)| = |\beta(p)| = p^{11/2}$. If there were a projective smooth variety X over $\mathbf{F}_p$ such that the polynomial $1-\tau(p)\cdot T+p^{11}T^2$ divided $P_{11}(X/\mathbf{F}_p, T)$, then the Riemann hypothesis for X would imply the Ramanujan conjecture for $\tau(p)$. The search for this X was carried out by Eichler, Shimura, Kuga, and Ihara [cf. Y. Ihara, Ann. of Math. (2) **85** (1967), 267–295; MR0207655; M. Kuga and G.

Shimura, ibid. (2) **82** (1965), 478–539; MR0184942]. They constructed an X which "should have worked", but because their X was not compact and had no obvious smooth compactification, its polynomial P_{11} did not necessarily have all its roots of the correct absolute value. The author then showed how to compactify their X and how to see that the Hecke polynomial $1 - \tau(p)T + p^{11}T^2$ divided a certain factor of P_{11}, the roots of which factor would have the "correct" absolute values if the Weil conjectures were true. Thus the truth of the Ramanujan conjecture became a consequence of the universal truth of the Riemann hypothesis for varieties over finite fields.

In 1939 R. A. Rankin [Proc. Cambridge Philos. Soc. **35** (1939), 351–372; MR0000411; correction, MR **1**, p. 400] had obtained the then-best estimate for $\tau(n)$ (namely $\tau(n) = O(n^{29/5})$) by studying the poles of the Dirichlet series $\sum(\tau(n))^2 \cdot n^{-s}$. R. P. Langlands [*Lectures in modern analysis and applications, III*, pp. 18–61, Lecture Notes in Math., Vol. 170, Springer, berlin, 1970; MR0302614] pointed out that the idea of Rankin's proof could easily be used to prove the Ramanujan conjecture, provided one knew enough about the location of the poles of an infinite collection of Dirichlet series formed from Δ by forming even tensor powers: for each even integer $2n$ one needed to know the poles of the function represented by the Euler product $\prod_p \prod_{i=0}^{2n} (1/(1-\alpha(p)^i \beta(p)^{2n-i} p^{-s}))^{\binom{2n}{i}}$.

The author studied Rankin's original paper in an effort to understand the remarks of Langlands. He realized that for L-series over curves over finite fields (instead of L-series over Spec($\mathbf{Z}$)), Grothendieck's cohomological theory [A. Grothendieck, Séminaire Bourbaki, Vol. 1964/1965, Exp. No. 279, facsimile reproduction, Benjamin, New York, 1966; see MR **33** #54201] of such L-series together with the Každan-Margulis monodromy result gave an a priori hold on the poles: Rankin's methods could therefore be combined with Lefschetz pencil-monodromy techniques to yield the Riemann hypothesis for varieties over finite fields, and with it the Ramanujan-Petersson conjecture as a corollary.

(III) **Other Applications**: Another arithmetic application is the estimation of exponential sums in several variables. Though technically difficult, the idea goes back to Weil [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR0027006], who showed how the Riemann hypothesis for curves over finite fields gave the "good" estimate for exponential sums in one variable.

As for geometric applications, we have already mentioned the hard Lefschetz theorem which is promised for the sequel to the present paper. There is also a whole chain of ideas built around the "yoga of weights", Grothendieck's catchphrase for deducing results on the cohomology of arbitrary varieties by assuming the Riemann hypothesis for projective non-singular varieties over finite fields. The whole of the author's "mixed Hodge theory" for complex varieties [Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57; "Théorie de Hodge, III", to appear in Inst. Hautes Études Sci. Publ. Math.], developed before his proof of the Riemann hypothesis, is intended to prove results about the cohomology of these varieties which follow from the Riemann hypothesis and from the systematic application of Hironaka's resolution of singularities. The recent work of the author, Griffiths, Morgan and Sullivan on the rational homotopy type of complex varieties is also considerably clarified by the use of the Riemann hypothesis.

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Deligne, Pierre

La conjecture de Weil. II. (French)

Institut des Hautes Études Scientifiques. Publications Mathématiques (1980), *no.* 52, 137–252.

This paper is the sequel to an earlier one by the author [same journal No. 43 (1974), 273–307; MR0340258]. By an essentially new method, which he calls the method of Hadamard-de la Vallée-Poussin because it involves showing that a large class of L-functions do not vanish on the line $\mathrm{Re}\, s = 1$, the author generalizes his results on the absolute values of the eigenvalues of Frobenius acting on étale cohomology to the case of twisted coefficients. He applies his results to prove the hard Lefschetz theorem and the local invariant cycle theorem, as well as to the study of the homotopy type of an algebraic variety.

Let X_0 be an algebraic variety defined over a finite field $\mathbf{F}_q$, and let X be the base extension of X_0 to the algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$. (The convention of dropping a subscript 0 to indicate base extension to $\mathbf{F}$ will be used without comment throughout this review.) Let x be a geometric point of X. The fundamental group $\pi_1(X_0, x)$ is an extension of $\hat{\mathbf{Z}}$ by $\pi_1(X, x)$. A smooth l-adic sheaf on X_0 [resp. Weil sheaf on X_0] is given by a continuous representation of $\pi_1(X_0, x)$ [resp. of $W(X_0, x) =$ subgroup of $\pi_1(X_0, x)$ mapping to $\mathbf{Z} \subset \hat{\mathbf{Z}}$] on a finite-dimensional vector space V over a finite extension of $\mathbf{Q}_l$.

Given a closed point $y \in X_0$, there is a conjugacy class $[F_y] \subset \pi_1(X_0, x)$ associated to the inverse of the Frobenius in $\mathrm{Gal}(\mathbf{F}/\mathbf{F}_q(y))$. A Weil sheaf E_0 on X_0 is i-pure of weight n for a given isomorphism i of $\overline{\mathbf{Q}}_l$ onto $\mathbf{C}$ if for all $y \in X_0$ the eigenvalues of F_y all have absolute value $q^{n/2}$. E_0 is pure of weight n if it is i-pure of weight n for any i. For example, if $f_0 \colon Y_0 \to X_0$ is smooth and projective, $R^m f_* \mathbf{Q}_l$ is pure of weight m by the Weil conjectures. E_0 is mixed if it is an iterated extension of pure sheaves.

For E_0 on X_0 a Weil sheaf, the cohomology $H^*_c(X, E)$ inherits a $\mathbf{Z}$-action and hence a notion of i-weights (computed for the inverse of the canonical generator of $\mathbf{Z}$). The main result in the paper under review is that E_0 i-mixed of weights $\leq n$ implies that $H^r_c(X, E)$ has i-weights $\leq n + r$.

To see the power of this result, suppose X_0 is an open smooth curve and that E_0 and G_0 are pure of weights n and m with $n \leq m$. Using the duality between compactly supported and ordinary cohomology, $H^1(X, \mathrm{Hom}(E, G))$ is seen to have weights ≥ 1. In particular there are no Frobenius invariants, hence no extensions of E_0 by G_0 nonsplit over X. This semisimplicity result, applied with $X_0 \subset \mathbf{P}^1$ parametrizing smooth members of a Lefschetz pencil on a variety V_0, and E the sheaf of middle dimensional cohomology groups on the fibres of the pencil, yields $E = E^\pi \oplus W$ with $\pi = \pi_1(X)$. W has no π invariants or coinvariants so E and W are perpendicular under the intersection pairing. This is equivalent to the classical assertion that any invariant vanishing cycle is trivial, and the hard Lefschetz theorem follows.

Let $Z(E_0, t) = \prod_y \det(I - F_y t | E_y)^{-1}$. The Grothendieck cohomological formula gives $Z(E_0, t)$ as a product $\det(I - Ft | H^i_c(X, E)^{(-1)^i})$. The proof of the main theorem is reduced to the case when E_0 has i-weight 0 and X_0 is an open curve. By duality it suffices to show the weights of $H^i_c(X, E) \leq 1$. An elementary argument based on the convergence of the infinite product for $Z(E_0, t)$ shows that these weights ≤ 2.

One assumes inductively the weights $\leq 1 + 2^{-k}$, and one considers $E_0 \boxtimes E_0$ on $X_0 \times X_0$. If $Y_0 \subset X_0 \times X_0$ is a hyperplane section one shows that the weights on $H^1(Y, E_0 \boxtimes E_0|Y)$ are integral and strictly less than 2. (It is here that the Hadamard-de la Vallée-Poussin method is used.) Fibering $X_0 \times X_0$ by a Lefschetz pencil, the above is sufficient to show the weights on $H^2(X \times X, E \boxtimes E) \leq 2 + 2^{-k}$, which gives $1 + 2^{-k-1} \geq$ weights of $H^1(X, E)$.

Let ω_s be the character $q^{-\deg(x)\cdot s}$ on $W = W(X_0, \overline{x})$, where $s \in \mathbf{C}$ and $\deg\colon W \to \mathbf{Z}$. The Hadamard-de la Vallée-Poussin idea is based on considering L-functions $L(\tau\omega_s)$ where X_0 is a curve and τ is a unitary representation of W. Let $\nu(\tau)$ be the residue at $s = 1$ of

$$\frac{-L'}{L}(\tau\omega_s) = \sum_{n,x} \log N(x) \cdot \mathrm{Tr}(\tau(F_x^n))\mathrm{N}(x)^{-ns}.$$

One knows that $\nu(\tau)$ is defined, $\nu(1) = 1$, $\nu(\tau) = \nu(\overline{\tau})$, and $\nu(\tau) \leq 0$ for $\tau \neq 1$. Extending ν to the Grothendieck group of virtual unitary representations by additivity and observing the terms on the right above are positive for s real and $\mathrm{tr}(\tau) > 0$, one also has $\nu(\rho \otimes \overline{\rho}) \geq 0$ for any virtual unitary representation ρ. The author proves a general lemma valid for any group W to the effect that such a function ν on the category of virtual unitary representations necessarily satisfies $\nu(\tau) = 0$ for τ irreducible unitary except $\tau = 1$ and possibly one other τ defined by a character of order 2. In the case at hand, such an exotic τ would correspond to a curve (double cover of X_0) whose zeta function had no pole at $s = 1$, and this cannot occur.

By a curious misprint, the running head throughout the paper is “la conjoncture de Weil”. More appropriate might have been “la conjunction de Weil et Deligne”.

Spencer J. Bloch

From MathSciNet, March 2018

MR0023093 (9,303c) 14.0X

Weil, André

Foundations of Algebraic Geometry. (English)

American Mathematical Society Colloquium Publications, vol. 29.
American Mathematical Society, *New York*, 1946, xix+289 pp.

Advances in the more arithmetic branches of modern algebra and their application to number theory naturally lead, as we may venture to say today, to problems which to the well-informed mathematician either appeared familiar as part of the heritage of classical algebraic geometry or seemed to be intrinsically adapted to a solution by more conceptual geometric methods. Furthermore, since major parts of the theory of algebraic functions of one variable had been fitted into the system of algebra it was sensible that similar interpretations and attempts at solutions were (and had to be) tried for higher dimensional problems. In order to understand and appreciate the ultimate significance of this book the reader may well keep in mind the preceding twofold motivation for the interest in algebraic geometry. Classical algebraic geometry made free use of a type and mode of reasoning with which the modern mathematician often feels uncomfortable, though the experience based on a rich and intricate source of examples made the founders of this discipline avoid serious mistakes in final results which lesser men might have been prone to make. The main purpose of this treatise is to formulate the broad principles of the intersection theory for algebraic varieties. We find those fundamental facts without which, for example, a good treatment of the theory of linear series would be difficult. The doctrine of this book is that an unassailable foundation (and thereby justification) of the basic concepts and results of algebraic geometry can be furnished by certain elementary methods of algebra. Thus, the reader will agree after some time that he is finding a delicate tool which can serve him to remove the traces of insecurity which occasionally accompany geometric reasoning. Incidentally, the term "elementary" used here and by the author is to be understood in a restricted technical sense, in the sense that general ideal theory and the theory of power series rings are not brought into play too often. The proofs require the general plan of using the "principles of specialization," as formulated algebraically by van der Waerden; and they are by no means elementary in the customary connotation. To some readers the adherence to a definite type of approach, where another author may have deemed it more instructive or appropriate to use slightly different methods, may tend to cloud occasionally immediate understanding by the less adept. However, once the reader has grasped the real geometric meaning of a definition or theorem (he then has to forget occasionally the fine points resulting from the facts that the author imposes no restriction on the characteristic of the underlying field of quantities) he will recognize how skilfully the language and methods of algebra are used to overcome certain limitations of spatial intuition.

The author begins his work with judiciously selected results from the theory of algebraic and transcendental extensions of fields [chapter I, Algebraic preliminaries]. Special emphasis has to be placed on inseparable extensions, which incidentally means a more complete account than is found in books on algebra. The further plan of the book is perhaps best appreciated if one starts to ponder over a more or less heuristic definition of "algebraic variety," and then asks one's self informally how one should define "intersections with multiplicities" of "subvarieties." Then, in view of the principle of local linearization in classical analysis, the author's arrangements of topics is more or less dictated by the ultimate subject under discussion, provided one does not place the interpretation of geometrical concepts by ideal theory at the head of the discussion. Therefore the technical definitions of point, variety, generic point and point set attached to a variety [chapter IV, The geometric language] must be preceded by suitable algebraic preparations [essentially in chapter II, Algebraic theory of specializations] and more arithmetic studies [chapter III, Analytic theory of specializations]. Crucial results in this connection, based on arithmetical considerations, are found in proposition 7 on page 60 and theorem 4 on page 62, where the existence of a well-defined multiplicity is proved for specializations. For further work, the author next introduces the concept of simple point of a variety in affine space by means of the linear variety attached to the point. [See the significant propositions 19 to 21 on pages 97–99.] Next, the intersection theory of varieties in affine space is presented through the following stages of increasing complexity: (i) intersection with a linear subspace of complementary dimension, the 0-dimensional case, with the important criterion for multiplicity 1 in proposition 7 on page 122, and ultimately the criterion for simple points in theorem 6 on page 136; (ii) intersection with a linear subspace of arbitrary dimension, with theorem 4 on page 129 which justifies the invariant meaning of the term "intersection multiplicity of a variety with a linear variety along a variety" [chapter V, Intersection multiplicities, special case]. In chapter VI, entitled General intersection theory, the results for the linear case are extended so as to culminate in the important theorem 2 on page 146 concerning the proper components of the intersection of two subvarieties in a given variety. Furthermore, all important properties of intersection multiplicities are established. Later, in appendix III, it is shown that the properties established for a certain symbol are characteristic for intersection multiplicities and uniquely define that concept. It may be mentioned that the topological definition of the chain intersections on manifolds coincides with the algebraically defined concept of this book. Of course, the underlying coefficient field has to be the field of all complex numbers and further simplifying assumptions on the variety have to be made. However, this comparison cannot be made at the level of chapters V and VI, since there one deals with affine varieties to which the ordinary topological considerations are not directly applicable.

The subsequent chapter VII, Abstract varieties, provides the necessary background for the aforementioned connections and also contains complete proofs of those results which one might have formulated first had one deliberately adopted ideal-theoretic intentions at an early stage. The abstract varieties of this chapter are obtained by piecing together varieties in affine spaces by means of suitably restricted birational transformations. This definition of the author has turned out to be very fruitful for the work on the Riemann hypothesis for function fields and the study of Abelian varieties in general. In the course of the work, the results of the preceding chapters are extended so as to lead up to the important theorem 8 on page 193 related to Hopf's "inverse homomorphism." The chapter ends with a theory of cycles of dimension s, that is, formal integral combinations of simple abstract subvarieties of dimension s. The notion of the intersection product of cycles is also introduced here [page 202], by means of which the investigation of equivalence theories can be initiated. This is done more explicitly in chapter IX, Comments and discussion; apparently the Riemann-Roch theorem for surfaces should now be accessible to a careful re-examination. As a further result, the theory of quasi-divisibility of Artin and van der Waerden is developed in theorems 3 and 4 on pages 224–225 and theorem 6 on page 230. These theorems exhibit the relations between the theory of cycles of highest dimension and the theory of quasi-divisibility, where naturally some of the results in appendix II, Normalization of varieties, are to be added for the necessary integral closure of the required rings of functions. In this appendix the author relates his results on the normalization of algebraic varieties to those of Zariski. At this point the individual reader may well compare the elementary and the ideal-theoretic approach to a group of theorems. In appendix I, Projective spaces, often used properties and facts concerning projective spaces are quickly developed on the basis of the preceding work. This brief discussion not only deals with results which are generally useful in algebraic geometry, but also contains one of the theorems on linear series of divisors which was frequently used in the classical work [see page 266]. Because of the wealth of material and the excellent "advice to the reader" prefacing this rich and important book the reviewer feels that he should mention some of the highlights and not delve into a discussion of technical details. In short, the only way to appreciate this treatise is actually to read it.

O. F. G. Schilling
From MathSciNet, March 2018

MR0027151 (10,262c) 14.0X
Weil, André
Sur les courbes algébriques et les variétés qui s'en déduisent. (French)
Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg **7** (1945).
Hermann et Cie., *Paris*, 1948, iv+85 pp.

Suppose that Ω_k is a field of algebraic functions of one variable with a finite field k of q elements for a coefficient field. Let p be a prime divisor of Ω_k which is trivial on k, i.e., a homomorphism of a suitable subring of Ω_k upon an algebraic extension $k(p)$ of k with $[k(p)\colon k] = d(p)$. Then the zeta function of Ω_k can be defined as $Z(u) = \prod_p (1-u^{d(p)})^{-1}$, where u is a complex variable, and the product is extended over all distinct prime divisors. Artin formulated the analogue of the Riemann hypothesis as the statement that the zeros of $Z(u)$ lie on the circle $|u| = q^{-\frac{1}{2}}$. The author gives in this paper the first complete proof of this Riemann hypothesis for function fields of arbitrary genus. His proof depends on a reformulation of the hypothesis as an assertion on the positiveness of a quadratic form [corollary 3 on page 70]. This quadratic form is derived from a trace function σ acting on a subring of the ring of correspondences of Ω_k. Thus the author requires a complete treatment of the theory of correspondences, and the major portion of the present paper is devoted to it.

For the unity of method and in order to establish a clearcut connection with the original papers of Castelnuovo, Enriques, Severi and others, it is found convenient to discuss "curves Γ over k" instead of the function field Ω_k. The concept of "curve" as used by the author requires a careful explanation as given in his book "Foundations of Algebraic Geometry" [Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; MR0023093], where it is not required that k be a finite field. (Naturally the author shows how the connection can be made between his theory and the slightly different theory of the field Ω_k.) For the treatment of the algebraic geometry on a curve the author permits himself to draw freely upon his book. Thus in the definition of the canonical divisors and in the proof of the theorem of Riemann-Roch the geometry of the 2-fold product of Γ by itself is used, and thereby tools like the intersection product, etc., are employed. Such tools are not absolutely necessary if one just desires to demonstrate the theorem of Riemann-Roch. However, for the discussion of the all-important trace function it is preferable to have available, for example, the author's definition of a canonical divisor [theorem 8 on page 42]. The correspondences on Γ are introduced as divisors on $\Gamma \times \Gamma$, and the uniqueness of the product of correspondences is established [theorem 6 on page 35] leading to a simple formula for the latter. This work and the discussion of the additive group of correspondences with the concept of equivalence require a comprehensive part of the general theory of cycles and of the intersection product in chapters VII and VIII of the author's book.

The ring of correspondences $\mathfrak{A} = \{\xi, \cdots\}$ is introduced by means of the equivalence relation and a symmetry operator $\xi \to \xi'$ is defined on it essentially by the interchange of the factors in $\Gamma \times \Gamma$; the trace $\sigma(\xi)$ is defined by means of intersection multiplicities. The function σ has all the formal properties of a trace [theorem 7 on page 41]. The most complicated part of the paper consists in proving that $\sigma(\xi\xi') > 0$ for nonzero ξ. Application of this basic inequality yields the positiveness of the quadratic form $\sigma(\xi\xi') = 2gx^2 + 2\sigma(\iota^n)xy + 2gq^n y^2$, where $\xi = x\delta + y\iota^n$ with integers x, y. In this formula g denotes the genus of Γ, δ denotes the identity correspondence and ι is the class of the correspondence $I(P) = P^\omega$, for each point P of Γ, where ω is the normalized automorphism of the algebraic completion $\overline{k}$ of k, $a^\omega = a^q$ for a in $\overline{k}$. The points P of Γ are essentially the same thing as the prime divisors of the coefficient extension $\Omega_k\overline{k}$. To each P there belongs, relative to k, a smallest field of definition $k(P)$. Then $d\log Z(u) = \sum_{n=1}^\infty \nu_n u^n du/u$, where ν_n is the number of distinct points P_j with $k(P_j) \subseteq k_n$ for the extension k_n of degree n over k. The numbers ν_n are identified with the numbers of components of certain intersection cycles related to $P \to P^{\omega^n}$. Thus the author shows $\nu_n = 1 + q^n - \sigma(\iota^n)$ and the expansion $d\log[(1-u)(1-qu)Z(u)] = \sum_{n=1}^\infty \sigma(\iota^n)u^n du/u$ together with the positiveness of $\sigma(\xi\xi')$ imply the Riemann hypothesis by means of a simple argument on analytic functions.

In the last paragraph further consequences of the structure of the ring $\mathfrak{A}$ are developed. Implications on the theory of L-series of function fields are given, and the connections with the groups of Hilbert and Artin's theory of the conductor are established. The higher ramification groups are given an interesting definition by means of the multiplicity of a point in a transformed cycle. These results depend on the identification of σ with the trace of a matrix representation of $\mathfrak{A}$ in a field of characteristic 0. The author announces the early publication of the pertinent facts which depend on the structure of the class group of Ω_k and further properties of the Jacobian variety attached to Γ, which incidentally already had to be used for the proof of the positiveness of $\sigma(\xi\xi')$ [pages 49–53].

O. F. G. Schilling

From MathSciNet, March 2018

MR0292838 (45 #1920) 14C25; 14F20

Kleiman, S. L.

Algebraic cycles and the Weil conjectures. (English)

Dix exposés sur la cohomologie des schémas, 359–386, *Adv. Stud. Pure Math., 3*, *North-Holland, Amsterdam*, 1968.

In a famous article [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393] A. Weil introduced the zeta function of a smooth projective variety, defined over the finite field k_q with q elements. This function $Z(t)$ is determined by the equation $\log Z(t) = \sum_{s=1}^\infty N_s t^s/s$, $N_s =$ number of k_{q^s}-valued points of V, and is an analogue of a number-theoretic nature for the arithmetic of the variety V of the Riemann zeta function for $\mathbf{Q}$. Hence, one can ask if $Z(t)$ has the same properties as the Riemann zeta function. The main questions are: (1) Does $Z(t)$ satisfy a functional equation? (2) Does the Riemann hypothesis hold for $Z(t)$? (3) Is $Z(t)$ rational? Weil pointed out in his article that the statements (1)-(3) would be true if there existed a cohomology theory with coefficients in a field of characteristic 0 for algebraic varieties that are defined over an algebraically closed field of characteristic $p > 0$, such that certain statements (like the Künneth formula or duality, etc.) are satisfied. Today the l-adic étale cohomology has been developed. In studying this l-adic cohomology the author realized that the Weil conjectures, i.e., the statements (1)-(3), are formal consequences of certain conjectures on algebraic cycles on algebraic varieties. These interesting conjectures arise from classical cohomology theory for smooth compact algebraic varieties over the complex numbers $\mathbf{C}$ and the classical work of Lefschetz and Hodge.

There is hope of attacking the Weil conjectures successfully by proving these conjectures on cycles. The author introduces in his article the notion of a Weil cohomology (the l-adic étale cohomology is one) and formulates for such a cohomology the conjectures of Lefschetz and Hodge type. These conjectures are known to be true for the classical cohomology. Also D. I. Lieberman has shown in his papers [Amer. J. Math. **90** (1968), 366–374; MR0230336; ibid. **90** (1968), 1165–1199; MR0238857] that the conjectures of Lefschetz type are true for abelian varieties The author then shows how to get the Weil conjectures from these conjectures. Furthermore, many interesting theorems are proved that relate the conjectures to each other.

H. Popp

From MathSciNet, March 2018

MR0269663 (42 #4558) 14F40; 14F30

Grothendieck, A.

Crystals and the de Rham cohomology of schemes.

Dix exposés sur la cohomologie des schémas, 306–358, *Adv. Stud. Pure Math.*, 3, *North-Holland, Amsterdam*, 1968.

Let X be a non-singular algebraic variety over the complex number field $\mathbf{C}$, and let X^{an} be the associated analytic manifold. The author has previously shown [Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 95–103; MR0199194] that the good classical cohomology $H^*(X^{\mathrm{an}}, \mathbf{C})$ with complex coefficients can be defined purely algebraically as the hypercohomology of the complex of sheaves of algebraic Kähler differential forms $(\Omega^{\cdot}_{X/\mathbf{C}}, d)$, the so-called de Rham cohomology $H_{DR}{}^*(X/\mathbf{C})$.

If X is singular, however, the complex $\Omega^{\cdot}_{X/\mathbf{C}}$, and hence the de Rham cohomology, is not good enough [T. Bloom and M. Herrera, Invent. Math. **7** (1969), 275–296; MR0248349].

When we define the de Rham cohomology of an algebraic variety over a field of characteristic p in a similar way, it has the serious drawbacks of having characteristic p and usually being too big.

To prove the remaining Riemann hypothesis part of the Weil conjecture for the congruence zeta functions [A. Weil, Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393; see also S. Kleiman, *Dix exposés sur la cohomologie des schémas*, pp. 359–386, North-Holland, Amsterdam, 1968], however, it is desirable to have a good p-adic cohomology for algebraic varieties of characteristic p, as well as the l-adic etale cohomology we already have [*Cohomologie étale des schémas* (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1963/64), Fasc. 1, 2, Inst. Hautes Études Sci., Paris, 1964]. P. Monsky and G. Washnitzer have a theory in that direction [Ann. of Math. (2) **88** (1968), 181–217; MR0248141; Monsky,. ibid. (2) **88** (1968), 218–238; MR0244272].

In these notes, the author proposes a new approach to overcome these drawbacks of the de Rham cohomology. As for the first, he defines the infinitesimal site [the stratifying site] for a scheme X over another S. The "open set" is a pair (U,T) of a Zariski open set of X and a nilpotent S-immersion $U \to T$ [nilpotent S-immersion with local retraction to X]. These define ringed topos, and the respective cohomology with values in the "structure sheaf" is proved to coincide with the de Rham cohomology ${H_{DR}}^*(X/S)$ when X/S is smooth and S is of characteristic 0. He conjectures that they give good alternatives also in the singular case in characteristic 0.

As for the second drawback in characteristic p, he proposes the crystalline topos and the connecting topos, which, respectively, coincide with our previous topos in characteristic 0. The proposal has been successfully carried out by P. Berthelot [C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A297-A300; MR0246882; ibid. **269** (1969), A357-A360; MR0249441; ibid. **269** (1969), A397-A400; MR0263833]. We now have a seemingly good p-adic cohomology.

When a morphism of schemes $f\colon X \to S$ is smooth, the relative version of the de Rham cohomology ${\mathcal{H}_{DR}}^*(X/S)$, which is a sheaf on S obtained as the hyperderived functor $\mathbf{R}f_*(\Omega^{\cdot}_{X/S})$, is proved to have a canonical absolute connection, the Gauss-Manin connection, in the sense of derived categories. When, moreover, S is smooth (hence X is), the connection exists in the ordinary sense. For an elementary proof of the existence and the integrability of the Gauss-Manin connection in the smooth case, see N. M. Katz and the reviewer [J. Math. Kyoto Univ. **8** (1968), 199–213; MR0237510] and Katz ["Nilpotent connections and the monodromy theorem: Applications of a result of Turritin" (Seminar on Degeneration of Algebraic Varieties), pp. 1–101, mimeographed notes, Inst. for Advanced Study, Princeton Univ., Princeton, N.J., 1970].

T. Oda
From MathSciNet, March 2018

MR1376246 (97e:14030) 14F99; 14C25, 14F20, 19E15

Suslin, Andrei; Voevodsky, Vladimir

Singular homology of abstract algebraic varieties.

Inventiones Mathematicae (1996), *no.* 1, 61–94.

In one of his most influential papers, A. Weil [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393] proved the "Riemann hypothesis for curves over functions fields", an analogue in positive characteristic algebraic geometry of the classical Riemann hypothesis. In contemplating the generalization of this theorem to higher-dimensional varieties (subsequently proved by P. Deligne [Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 273–307; MR0340258] following foundational work of A. Grothendieck), Weil recognized the importance of constructing a cohomology theory with good properties. One of these properties is functoriality with respect to morphisms of varieties. J.-P. Serre showed with simple examples that no such functorial theory exists for abstract algebraic varieties which reflects the usual (singular) integral cohomology of spaces. Nevertheless, Grothendieck together with M. Artin [*Théorie des topos et cohomologie étale des schémas. Tome 1*, Lecture Notes in Math., 269, Springer, Berlin, 1972; MR0354652; *Tome 2*, Lecture Notes in Math., 270, 1972; MR0354653; *Tome 3*, Lecture Notes in Math., 305, 1973; MR0354654] developed étale cohomology which succeeds in providing a suitable Weil cohomology theory provided one considers cohomology with finite coefficients (relatively prime to residue characteristics). This theory, presented in J. Milne's book [*Étale cohomology*, Princeton Univ. Press, Princeton, N.J., 1980; MR0559531], relies on a new formulation of topology and sophisticated developments in sheaf theory.

In the present paper, the authors offer a very different solution to the problem of providing an algebraic formulation of singular cohomology with finite coefficients. Indeed, their construction is the algebraic analogue of the topological construction of singular cohomology [see, e.g., E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966; MR0210112], thereby being much more conceptual. Their algebraic singular cohomology with (constant) finite coefficients equals étale cohomology for varieties over an algebraically closed field. The proof of this remarkable fact involves new topologies, new techniques, and new computations reminiscent of the earlier work of Artin and Grothendieck.

To understand the authors' construction, we recall the classical theorem of A. Dold and R. Thom [Ann. of Math. (2) **67** (1958), 239–281; MR0097062]. This asserts that the singular homology of a CW complex X is naturally isomorphic to the homotopy groups of the simplicial abelian group $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^dX))^+$, the group completion of the singular complex of the topological abelian monoid $\coprod_{d\geq 0} S^dX$. Now, if X is an algebraic variety, so are its symmetric products. Moreover, homotopy groups of the simplicial abelian group $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^dX))^+$ can be computed as the homology of the associated chain complex, which we denote by $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^dX))^{\sim}$. The construction of Suslin-Voevodsky, first proposed by Suslin in a talk in Luminy in 1987, is to replace the singular complex by its algebraic analogue. Algebraic singular simplices were exploited years ago by M. Karoubi and O. Villamayor [C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A416–A419; MR0251717] and more recently used by S. Bloch in his formulation of higher Chow groups [Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815].

The fundamental theorem of Suslin-Voevodsky is that if X is an algebraic scheme of finite type over an algebraically closed field k of characteristic $p \geq 0$ and if n is an integer prime to p, then the étale cohomology of X with $\mathbf{Z}/n$ coefficients can be computed as $\mathrm{Ext}^*((\mathrm{Sing}_{\cdot}^{\mathrm{alg}}(\coprod_{d\geq 0} S^dX))^+, \mathbf{Z}/n)$. (The published statement assumes that the ground field k is of characteristic 0; as the authors soon realized, recent work of J. de Jong giving a weak version of resolution of singularities for varieties over fields of positive characteristic enables this extension to arbitrary characteristic.) Although the formulation of this theorem is relatively elementary, its proof involves sophisticated techniques of abstract algebraic geometry as well as insights from algebraic K-theory. Indeed, the authors encountered this theorem as a part of a sweeping approach to motivic cohomology and algebraic K-theory [see, e.g., V. Voevodsky, A. Suslin and E. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., to appear]. Underlying the authors' approach to (motivic) cohomology is the utilization of algebraic cycles. Maps from a normal variety S (e.g., a standard algebraic simplex Δ^k) to a symmetric product of X correspond to cycles on $S \times X$ finite and surjective over X.

The geometric heart of the proof is the authors' determination of the relative algebraic singular homology of a relative curve in terms of the relative Picard group, just as a key first ingredient for étale cohomology is the understanding of the étale cohomology of curves. This computation leads to a general form of the rigidity theorem of O. Gabber [in *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, 59–70, Contemp. Math., 126, Amer.

Math. Soc., Providence, RI, 1992; MR1156502] and H. A. Gillet and R. W. Thomason [J. Pure Appl. Algebra **34** (1984), no. 2-3, 241–254; MR0772059] which played a key role in Suslin's proof of the Quillen-Lichtenbaum conjecture for an arbitrary algebraically closed field [A. A. Suslin, J. Pure Appl. Algebra **34** (1984), no. 2-3, 301–318; MR0772065]. Namely, the authors consider homotopy invariant presheaves with transfers, a basic structure which now plays a central role in their approach to motivic cohomology. The example of most interest for the present work is the "free sheaf generated by X", whose values on standard algebraic simplices determine the chain complex $(\mathrm{Sing}^{\mathrm{alg}}_{\cdot}(\coprod_{d\geq 0} S^d X))^{\sim}$. This example fits their general context of presheaf with transfers thanks to the theorem that any "qfh-sheaf" admits the structure of a presheaf with transfers.

An essential ingredient in the authors' approach to cohomology is a further generalization of the étale topology in which proper maps arising in resolutions of singularities occur as coverings. Voevodsky's "h-topology" and its quasi-finite version leading to qfh-sheaves [cf. Selecta Math. (N.S.) **2** (1996), no. 1, 111–153] play an important role. The authors' rigidity theorem asserts the equality of various Ext-groups from sheaves associated to a homotopy invariant presheaf F with transfers to $\mathbf{Z}/n$, where these Ext-groups are computed in the étale topology and various topologies associated to the h-topology. Much of the formal effort in establishing their comparison theorems consists in analyses and manipulations of resolutions of sheaves for these topologies.

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