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ABOUT THE COVER: WARING'S PROBLEMS

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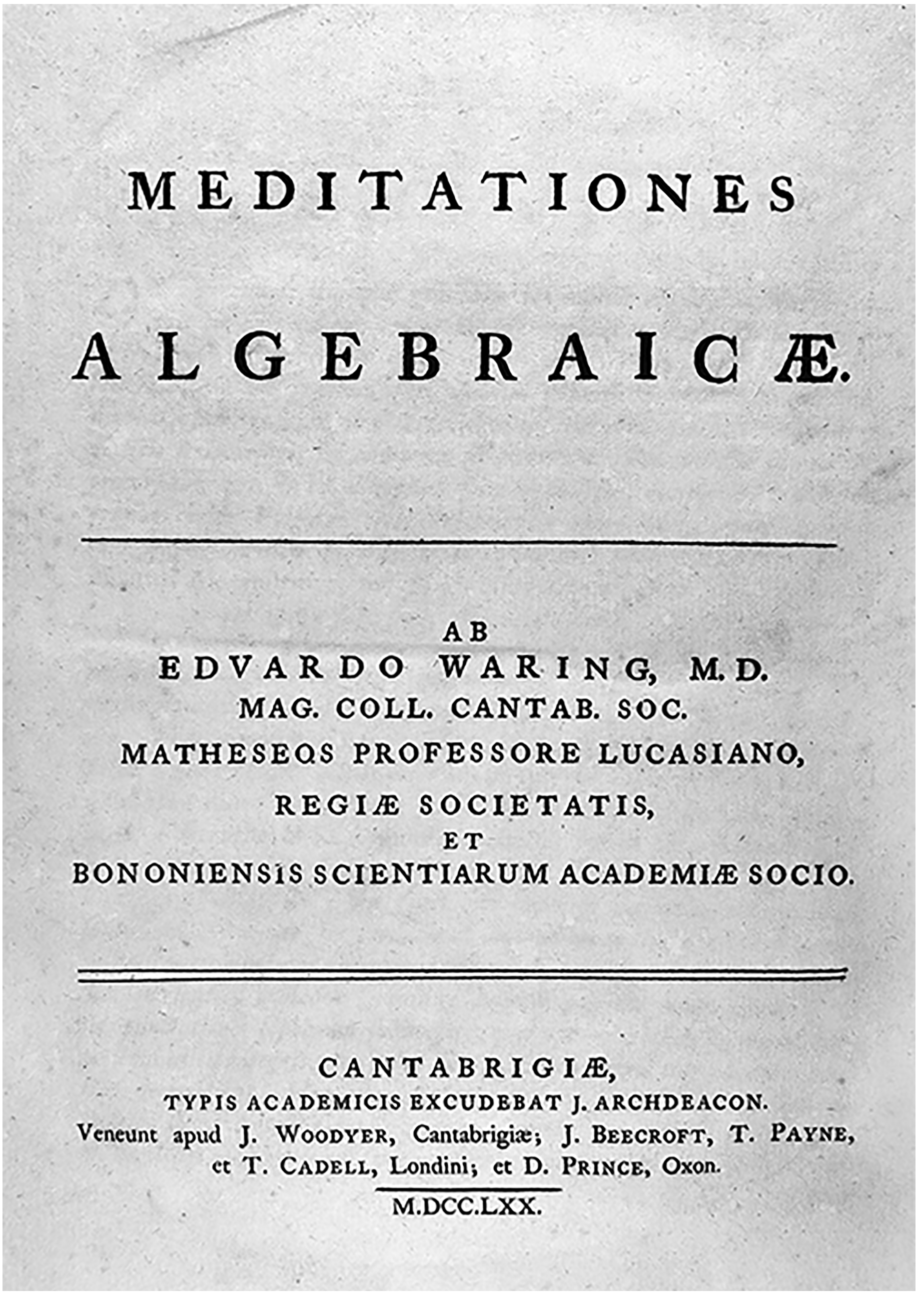
In some circles Edward Waring (1736–1798) is a household name. Some renowned mathematicians have spent good parts of their professional lives on questions that fall into the branch of number theory that includes Waring's Problem. But if you look at the title pages of Waring's books, there is something that leaps out at the reader (see Figure 1). Just what is it that looks strange here?

We see the usual identifying information on the author—that he was Lucasian Professor of Mathematics at Cambridge University (at the very young age of 24), a Fellow of the Royal Society at 27 (Newton had had to wait for that honor until he was 30), and also a member of the less obvious Academy of Sciences of Bologna. His affiliation at Cambridge was in one case given as Trinity College, but the more accurate was the claim that he was a fellow at Magdalene College. Having been a Lucasian Professor he was in good company, previous holders of this chair having been Isaac Barrow, John Colson, and Isaac Newton, with successors including Charles Babbage, G. G. Stokes, Paul Dirac, and Stephen Hawking. Further, as a student Waring had been Senior Wrangler, no small feat at Cambridge, and he won the Copley Medal in 1784. It is perhaps more interesting to note that he had a student who became famous among mathematics majors: John Wilson.

Waring is best known for his conjectures, though he never proved the two for which he is best known. The first is that every positive integer is a cube or the sum of at most nine positive cubes. Lagrange had earlier proved that every positive integer can be written as the sum of at most four squares, so Waring's conjecture was moving in a natural direction. He further conjectured that every positive integer can be expressed either as a fourth power of an integer or the sum of no more than 19 fourth powers. But he failed to publish proofs of these; that was left to others. He did a lot of conjecturing and some proving. (He was, apparently, the first to describe the ratio test for determining the convergence of an infinite series, a test first attributed to d'Alembert, but probably currently claimed for Cauchy.)



MEDITATIONES

ALGEBRAICÆ.

AB
EDVARDO WARING, M.D.
MAG. COLL. CANTAB. SOC.
MATHESEOS PROFESSORE LUCASIANO,
REGIÆ SOCIETATIS,
ET
BONONIENSIS SCIENTIARUM ACADEMIÆ SOCIO.

CANTABRIGIÆ,
TYPIS ACADEMICIS EXCUDEBAT J. ARCHDEACON.
Veneunt apud J. WOODYER, Cantabrigiæ; J. BEECROFT, T. PAYNE,
et T. CADELL, Londini; et D. PRINCE, Oxon.
M.DCC.LXX.

FIGURE 1. CLEAR COPY OF THE COVER. Title page of the first edition of Waring's *Meditationes algebraicæ*, 1770

What stands out incongruously on the title pages is that he is listed as "Eduardo Waring M.D." Really? The Lucasian Professor of Mathematics was a physician? Yes. He took a degree in medicine in mid-life and practiced medicine for some years, privately and in a local hospital, while he was still the Lucasian Professor at Cambridge. This was perhaps to the relief of his students, though not, perhaps, to his patients. Waring was someone who might now be described as an "odd duck", seldom being encouraged to deliver lectures because his listeners described them as badly organized and written out in illegible handwriting. (It could be noted that some of his successors in the teaching profession also exhibit similar tendencies.) He continued to write, though again, his readers have found his mathematical exposition difficult to follow. But his patients also found communicating with him difficult, so in later life he returned to doing mathematics full time, sometimes reorganizing his ideas in his subsequent books. His principal publications were *Miscellanea analytica* (1762), *Meditationes algebraicae* (1770, 1782), *Proprietates algebraicarum curvarum* (1772), and *Meditationes analyticae* (1776). Books by Waring appear very infrequently in the market, perhaps partly because they are difficult to read. Oddly enough, one of the authors of this note owns a copy of the second title on this list, one that was owned by G. H. Hardy that has his signature and the date he acquired it inscribed in his own hand on an opening endpaper—1 March 1919—and as well a copy of the third on this list, one owned successively by Hardy and J. E. Littlewood, both of whom signed the opening endpaper. The first was acquired from a London dealer in 1964, the second in Los Angeles in 1976. One can only speculate on how they happened to come on the market. Most of Hardy's papers went to the London Mathematical Society on his death, and other materials to his sister, G. E. Hardy. Both of these volumes probably became available to booksellers well after Hardy's death in 1947, even after his sister's death in 1963. These books have stamps indicating that they were deaccessioned in one case by Cambridge and, in the other, the University of London. How they travelled from Hardy's shelves to institutions to booksellers and eventually to California is anybody's guess at this stage.

Readers aware of Hardy and Littlewood's collaboration will recall that the Waring Problem took up many of their later years. As early as 1920, though, Hardy and Littlewood were writing about Waring's Problem [1]. If we define $g(k)$ to be the smallest number of powers that suffice to add up to any positive integer n, then as we noted above, $g(2) = 4$ (proved by Lagrange) and any whole number can be written as a sum of 9 or fewer cubes (that is, $g(3) = 9$). There are numbers that cannot be written as a sum of 8 cubes—23 is an example. Similarly it was shown early on that $g(4) = 19$, just as Waring had predicted. The question of finding $g(k)$ for any k proved to be difficult. A major work of David Hilbert in 1909 showed the existence of $g(k)$ for any $k > 2$. But his method was not constructive, so it was of no use in calculating other values of the function. There have been partial results, though largely in the form of bounds on the values.

Hardy and Littlewood transformed the problem to one of finding $G(k)$ where $G(k)$ is the smallest number of kth powers sufficient to write any positive integer as a sum of kth powers beyond a certain point. The tricky cases seem to come along fairly soon, that is, those integers that require more kth powers are usually comparatively small. For example, $g(4) = 19$ but $G(4) = 16$. The extra three

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meri in arithmeticâ progreſſione & quorum communes differentiæ haud reſpective diviſibiles ſint per numeros $1 \times 2 \times 3$, $1 \times 2 \times 3 \times 5$, $1 \times 2 \times 3 \times 5 \times 7$, $1 \times 2 \times 3 \times 5 \times 7 \times 11$, $1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13$, $1 \times 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$, &c. ni 3, 5, 7, 11, 13, 17, &c. ſint reſpective primi termini arithmeticæ ſeriei, in quibus caſibus poſſint eſſe ſolummodo 3, 5, 7, 11, 13, 17, &c. termini arithmeticarum ſerierum reſpective, quorum differentiæ prædictos diviſores haud admittant.

Hæc proprietas primorum numerorum facile demonſtrari poſſit, & plures ex eodem fonte hauriri poſſint.

Sit n numerus primus, & $\dfrac{1 \times 2 \times 3 \times 4 \ldots \overline{n-2} \times \overline{n-1} + 1}{n}$ erit integer numerus, e. g. $\dfrac{1 \times 2 + 1}{3} = 1$, $\dfrac{1 \cdot 2 \cdot 3 \cdot 4 + 1}{5} = 5$, $\dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1}{7} = 103$, &c. Hanc maxime elegantem primorum numerorum proprietatem invenit vir clariſſimus, rerumque mathematicarum peritiſſimus Joannes Wilſon Armiger.

Sed datâ aliquâ quantitate, cujus n ſit diviſor, facile deduci poſſint infinitæ aliæ, quarum eadem quantitas (n) erit diviſor: e. g. $\dfrac{1 \cdot 2 \cdot 3 \ldots \overline{n-2} - 1}{n}$, $\dfrac{1 \cdot 2^2 \cdot 3 \ldots \overline{n-3} + 1}{n}$, $\dfrac{1 \cdot 2^2 \cdot 3^2 \ldots \overline{n-4} - 1}{n}$ $\ldots \dfrac{2^2\, 3^2\, 4^2\, 5^2 \ldots \overline{\dfrac{n-1}{4}}^{\,2} \pm 1}{n}$ (ubi erit $+1$, quando $\dfrac{n-1}{2}$ ſit par numerus, ſin aliter -1) integri erunt numeri.

Demonſtrationes vero hujuſmodi propoſitionum eo magis difficiles erunt, quod nulla fingi poteſt notatio, quæ primum numerum exprimat.

SCHOLIUM.

Hactenus de algebraicis æquationibus, eadem etiam applicari poſſint ad ullas alias æquationes, quæ e ſubſtitutione transformari poſſint

FIGURE 2. A page from Waring's *Meditationes algebraicæ* with the first appearance of what we now call Wilson's theorem in the paragraph beginning "Sed data aliqua quantitate. . . ", which is attributed to Waring's student "John Wilson, Knight".

fourth powers are necessary to handle the smaller numbers, but for sufficiently large numbers 16 will do the job. The roster of major mathematicians contributing to these problems is very distinguished: Harold Davenport, Edmund Landau, S. Ramanujan, I. M. Vinogradov, L. E. Dickson, S. S. Pillai, I. M. Niven, Kurt Mahler, Ju. V. Linnik, D. R. Heath-Brown, K. Kawada, I. K. Hua, Carl Ludwig Siegel, Paul Erdős, D. J. Lewis, R. C. Vaughan, and T. D. Wooley, among many others. Most later research on Waring's Problem has been devoted to $G(k)$. A comprehensive survey of work over the past century has appeared in [2] and includes related problems, such as unlike powers, sums of prime powers, extensions to algebraic number fields, and so on. The authors Vaughan and Wooley have themselves contributed major results on Waring's conjectures. So the field is alive and well.

Aside from these problems Waring did make contributions to other questions: the classification of curves, systems of equations, and the study of symmetric functions with glimpses of what was to become group theory. Then there is another aspect of Waring's work that might explain the low esteem for him during his lifetime. He was probably on the "wrong" side of the Newton–Leibniz controversy, even though he was not born until 20 years after Newton died. His approach to many problems was more analytical than geometrical, so he looked to models on the Continent, influenced by Euler, d'Alembert, and Clairaut, opening up new directions in partial differential equations that had not been investigated earlier. Of course, being British he did refer to them as "partial fluxional equations".

We cannot leave the subject of Edward Waring without calling attention to the penultimate page of his 1770 *Meditationes algebraicæ* (see Figure 2), where we find the first appearance in print of what every student of elementary number theory knows as Wilson's theorem, here credited by Waring to his student Sir John Wilson (Joannes Wilson Armiger). In slightly different notation from the original, it says that for p a prime,

$$(p-1)! \equiv -1 \pmod{p}.$$

It was worth the wait.

References

[1] G. H. Hardy and J. E. Littlewood, *A new solution of Waring's problem*, Quart. J. Math. **48** (1920), 272–293.

[2] R. C. Vaughan and T. D. Wooley, *Waring's problem: a survey*, Number theory for the millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 301–340. MR1956283

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