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SELECTED MATHEMATICAL REVIEWS

related to the papers in the previous section by
MARC LEVINE AND DANIEL GRAYSON

MR1376246 (97e:14030) 14F99; 14C25, 14F20, 19E15
Suslin, Andrei; Voevodsky, Vladimir
Singular homology of abstract algebraic varieties.
Inventiones Mathematicae **123** (1996), *no.* 1, 61–94.

In one of his most influential papers, A. Weil [Bull. Amer. Math. Soc. **55** (1949), 497–508; MR0029393] proved the "Riemann hypothesis for curves over functions fields", an analogue in positive characteristic algebraic geometry of the classical Riemann hypothesis. In contemplating the generalization of this theorem to higher-dimensional varieties (subsequently proved by P. Deligne [Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 273–307; MR0340258] following foundational work of A. Grothendieck), Weil recognized the importance of constructing a cohomology theory with good properties. One of these properties is functoriality with respect to morphisms of varieties. J.-P. Serre showed with simple examples that no such functorial theory exists for abstract algebraic varieties which reflects the usual (singular) integral cohomology of spaces. Nevertheless, Grothendieck together with M. Artin [*Théorie des topos et cohomologie étale des schémas. Tome 1*, Lecture Notes in Math., 269, Springer, Berlin, 1972; MR0354652; *Tome 2*, Lecture Notes in Math., 270, 1972; MR0354653; *Tome 3*, Lecture Notes in Math., 305, 1973; MR0354654] developed étale cohomology which succeeds in providing a suitable Weil cohomology theory provided one considers cohomology with finite coefficients (relatively prime to residue characteristics). This theory, presented in J. Milne's book [*Étale cohomology*, Princeton Univ. Press, Princeton, N.J., 1980; MR0559531], relies on a new formulation of topology and sophisticated developments in sheaf theory.

In the present paper, the authors offer a very different solution to the problem of providing an algebraic formulation of singular cohomology with finite coefficients. Indeed, their construction is the algebraic analogue of the topological construction of singular cohomology [see, e.g., E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966; MR0210112], thereby being much more conceptual. Their algebraic singular cohomology with (constant) finite coefficients equals étale cohomology for varieties over an algebraically closed field. The proof of this remarkable fact involves new topologies, new techniques, and new computations reminiscent of the earlier work of Artin and Grothendieck.

To understand the authors' construction, we recall the classical theorem of A. Dold and R. Thom [Ann. of Math. (2) **67** (1958), 239–281; MR0097062]. This asserts that the singular homology of a CW complex X is naturally isomorphic to the homotopy groups of the simplicial abelian group $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^d X))^+$, the group completion of the singular complex of the topological abelian monoid $\coprod_{d\geq 0} S^d X$. Now, if X is an algebraic variety, so are its symmetric products. Moreover, homotopy groups of the simplicial abelian group $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^d X))^+$ can be computed as the homology of the associated chain complex, which we denote by $(\mathrm{Sing}_{\cdot}(\coprod_{d\geq 0} S^d X))^{\sim}$. The construction of Suslin-Voevodsky, first proposed by

Suslin in a talk in Luminy in 1987, is to replace the singular complex by its algebraic analogue. Algebraic singular simplices were exploited years ago by M. Karoubi and O. Villamayor [C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A416–A419; MR0251717] and more recently used by S. Bloch in his formulation of higher Chow groups [Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815].

The fundamental theorem of Suslin-Voevodsky is that if X is an algebraic scheme of finite type over an algebraically closed field k of characteristic $p \geq 0$ and if n is an integer prime to p, then the étale cohomology of X with $\mathbf{Z}/n$ coefficients can be computed as $\mathrm{Ext}^*((\mathrm{Sing}_\cdot^{\mathrm{alg}}(\coprod_{d\geq 0} S^d X))^+, \mathbf{Z}/n)$. (The published statement assumes that the ground field k is of characteristic 0; as the authors soon realized, recent work of J. de Jong giving a weak version of resolution of singularities for varieties over fields of positive characteristic enables this extension to arbitrary characteristic.) Although the formulation of this theorem is relatively elementary, its proof involves sophisticated techniques of abstract algebraic geometry as well as insights from algebraic K-theory. Indeed, the authors encountered this theorem as a part of a sweeping approach to motivic cohomology and algebraic K-theory [see, e.g., V. Voevodsky, A. Suslin and E. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., to appear]. Underlying the authors' approach to (motivic) cohomology is the utilization of algebraic cycles. Maps from a normal variety S (e.g., a standard algebraic simplex Δ^k) to a symmetric product of X correspond to cycles on $S \times X$ finite and surjective over X.

The geometric heart of the proof is the authors' determination of the relative algebraic singular homology of a relative curve in terms of the relative Picard group, just as a key first ingredient for étale cohomology is the understanding of the étale cohomology of curves. This computation leads to a general form of the rigidity theorem of O. Gabber [in *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, 59–70, Contemp. Math., 126, Amer. Math. Soc., Providence, RI, 1992; MR1156502] and H. A. Gillet and R. W. Thomason [J. Pure Appl. Algebra **34** (1984), no. 2-3, 241–254; MR0772059] which played a key role in Suslin's proof of the Quillen-Lichtenbaum conjecture for an arbitrary algebraically closed field [A. A. Suslin, J. Pure Appl. Algebra **34** (1984), no. 2-3, 301–318; MR0772065]. Namely, the authors consider homotopy invariant presheaves with transfers, a basic structure which now plays a central role in their approach to motivic cohomology. The example of most interest for the present work is the "free sheaf generated by X", whose values on standard algebraic simplices determine the chain complex $(\mathrm{Sing}_\cdot^{\mathrm{alg}}(\coprod_{d\geq 0} S^d X))^\sim$. This example fits their general context of presheaf with transfers thanks to the theorem that any "qfh-sheaf" admits the structure of a presheaf with transfers.

An essential ingredient in the authors' approach to cohomology is a further generalization of the étale topology in which proper maps arising in resolutions of singularities occur as coverings. Voevodsky's "h-topology" and its quasi-finite version leading to qfh-sheaves [cf. Selecta Math. (N.S.) **2** (1996), no. 1, 111–153] play an important role. The authors' rigidity theorem asserts the equality of various Ext-groups from sheaves associated to a homotopy invariant presheaf F with transfers to $\mathbf{Z}/n$, where these Ext-groups are computed in the étale topology and various topologies associated to the h-topology. Much of the formal effort in establishing their comparison theorems consists in analyses and manipulations of resolutions of sheaves for these topologies.

Eric M. Friedlander
From MathSciNet, July 2018

MR1648048 (99j:14018) 14F35; 14A15, 55U35
Voevodsky, Vladimir
$\mathbf{A}^1$-homotopy theory.
Proceedings of the International Conbress of Mathematicians, Vol. I (Berlin, 1998). *Documenta Mathematica* (1998), *no.* Extra Vol I, 579–604.

To this outside observer, one of the most significant strands in the recent history of algebraic geometry has been the search for good cohomology theories of schemes. Each new cohomology theory has led to significant advances, the most famous being étale cohomology and the proof of the Weil conjectures.

In a beautiful tour de force, Voevodsky has constructed all reasonable cohomology theories on schemes simultaneously by constructing a stable homotopy category of schemes. This is a triangulated category analogous to the stable homotopy category of spaces studied in algebraic topology; in particular, the Brown representability theorem holds, so that every cohomology theory on schemes is an object of the Voevodsky category. This work is, of course, the foundation of Voevodsky's proof of the Milnor conjecture.

The paper at hand is an almost elementary introduction to these ideas, mostly presenting the formal structure without getting into any proofs that require deep algebraic geometry. It is a beautiful paper, and the reviewer recommends it in the strongest terms. The exposition makes Voevodsky's ideas seem obvious; after the fact, of course.

One of the most powerful advantages of the Voevodsky category is that one can construct cohomology theories by constructing their representing objects, rather than by describing the groups themselves. The author constructs singular homology (following the ideas of A. Suslin and Voevodsky [Invent. Math. **123** (1996), no. 1, 61–94; MR1376246]), algebraic K-theory, and algebraic cobordism in this way. Throughout the paper, there are very clear indications of where Voevodsky thinks the theory needs further work, and the paper concludes with a discussion of possible future directions.

Mark Hovey
From MathSciNet, July 2018

MR1813224 (2002f:14029) 14F35; 19E08
Morel, Fabien; Voevodsky, Vladimir
$\mathbf{A}^1$-homotopy theory of schemes.
Institut des Hautes Études Scientifiques. Publications Mathématiques (1999), *no.* 90, 45–143.

Algebraic geometry has long thrived on the importation of ideas and principles from topology. However, before the constructions of the article under review, there was no framework available for a systematic lifting of the detailed techniques of algebraic topology to the realm of algebraic geometry. Now that this framework along with its extension to the theory of $\mathbf{P}^1$-spectra is available, we are beginning to see modern homotopy theory being used in its algebro-geometric version, giving new insights to such classical areas as quadratic forms and number theory.

The construction of Karoubi-Villamayor K-theory KV_* [M. Karoubi and O. Villamayor, Math. Scand. **28** (1971), 265–307 (1972); MR0313360] gave the first hint for what the proper framework for doing algebraic topology in the setting of algebraic geometry would be.

For a regular ring A, $K_1(A)$ is presented as the infinite general linear group $\mathrm{GL}_\infty(A)$ modulo the subgroup generated by the image of $\mathrm{GL}_\infty(A[T])$ under the map $g(T) \mapsto g(0)^{-1}g(1)$, that is, $K_1(A)$ is the group of "path components" of $\mathrm{GL}_\infty(A)$, where the affine line replaces the unit interval in defining a path.

One then applies a simplicial machinery to construct the rest of KV-theory. Specifically, one defines the algebraic n-simplex Δ^n as the hyperplane in affine $(n+1)$-space given by the equation $t_0 + \cdots + t_n = 1$. For a ring A, one has the ring $\Delta_n(A) := A[t_0, \ldots, t_n]/(\sum_i t_i - 1)$. The usual formulas for the coface and codegeneracy maps among the standard topological simplices are all linear in the coordinates t_i, so define a purely algebraic structure, the cosimplicial scheme Δ^*, or the simplicial ring $\Delta_*(A)$. For Karoubi-Villamayor K-theory, one replaces A with $\Delta_*(A)$ and applies GL_∞, forming the simplicial set $\mathrm{GL}_\infty(\Delta_*(A))$. $\mathrm{KV}_*(A)$ is defined as the homotopy groups of the geometric realization of this simplicial set: $\mathrm{KV}_n(A) := \pi_{n-1}(|\mathrm{GL}_\infty(\Delta^*(A))|)$.

A similar process occurs in S. Bloch's construction of the higher Chow groups [Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815], and in A. Suslin's construction of abstract homology [lecture at the conference "Les régulateurs" (Luminy, 1987); per revr.; see also A. A. Suslin and V. Voevodsky, Invent. Math. **123** (1996), no. 1, 61–94; MR1376246]. Both start with a functor defined via some generators (cycles or families of zero-cycles), with the relations for a variety X given by (essentially) the same generators on $X \times \mathbf{A}^1$, using the difference of the restriction to $X \times 0$ and $X \times 1$. They apply the "functor" of generators to $X \times \Delta^*$ (in Bloch's case, the fact that cycles do not really form a functor creates some technical difficulties, but we will pass over this point), and then take the homotopy groups of the associated simplicial set.

In all these constructions, the role of the unit interval in topology is being replaced with the affine line $\mathbf{A}^1$. Morel and Voevodsky build on this idea, creating a category in which one can perform the basic constructions used in topology, but where the building blocks are algebraic varieties instead of topological spaces, and where homotopy uses the affine line instead of the unit interval. The construction proceeds in three main steps.

The first step is to embed the category of algebraic schemes over a fixed base S into the category of sheaves (on S-schemes) for the Nisnevich topology, by sending a variety X to the representing sheaf $Y \mapsto \mathrm{Hom}_S(Y, X)$. Embedding into the category of sheaves enables one to perform many of the operations used in topology, most notably, the operation of taking the quotient of a topological space by a subspace. This is only rarely possible for algebraic varieties, but for sheaves it is a triviality. The next step is to introduce some "classical" topology by enlarging to the category of simplicial sheaves. One then has the notion of an $\mathbf{A}^1$-weak equivalence: A map $f\colon X \to Y$ is an $\mathbf{A}^1$-weak equivalence if f induces a stalk-wise weak equivalence of the simplicial Hom's, $f^*\colon \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, for all $\mathbf{A}^1$-local Z, where Z is

$\mathbf{A}^1$-local if the map on simplicial Hom's,

$$p_1^*\colon \mathrm{Hom}(W,Z) \to \mathrm{Hom}(W \times \mathbf{A}^1, Z),$$

is a stalk-wise weak equivalence for all W. One localizes the category of simplicial sheaves with respect to $\mathbf{A}^1$-weak equivalence, and the construction of the $\mathbf{A}^1$-homotopy category of S-schemes $\mathcal{H}(S)$ is complete.

One result of the paper is that this $\mathbf{A}^1$-localization can be accomplished by using Δ^*. If X is a simplicial sheaf, one has the simplicial sheaf $\mathrm{Sing}(X) := \mathrm{Hom}(\Delta^*, X)$. Ignoring some technicalities, the functor sending X to $\mathrm{Sing}^\infty(X)$ (the infinite iterate of the functor Sing) is equivalent to the $\mathbf{A}^1$-localization of the category of simplicial sheaves.

As an indication that the theory works as expected, Morel and Voevodsky show that, just as in topology, the classifying space $B\mathrm{GL}_\infty$ represents algebraic K-theory (for smooth S-schemes): $K_n(X) \cong \mathrm{Hom}_{\mathcal{H}(S)}(\Sigma^n X, B\mathrm{GL}_\infty)$.

The fun really begins with the next step beyond the article under review: the construction of the category of $\mathbb{P}^1$-spectra. This allows one, among other things, to recreate the whole world of extraordinary cohomology theories in the algebro-geometric setting, and gives the proper framework for Voevodsky's beautiful proof of the Milnor conjecture.

Marc Levine

From MathSciNet, July 2018

MR1744945 (2001g:14031) 14F42; 19D45, 19E15, 19E20

Suslin, Andrei; Voevodsky, Vladimir

Bloch-Kato conjecture and motivic cohomology with finite coefficients.

The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, *NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht*, 2000.

Let F be a field, m an integer prime to the characteristic of F, and Sm/F be the category of smooth schemes over F. The conjecture of Beilinson and Lichtenbaum for weight n states that the natural map $\mathbf{Z}/m(n) \to \tau_{\le n}\mathbb{R}\alpha_*\mu_m^{\otimes n}$ is a quasi-isomorphism. Here $\mathbf{Z}/m(n)$ is the mod m motivic complex, and $\alpha\colon (\mathrm{Sm}/F)_{\text{ét}} \to (\mathrm{Sm}/F)_{\mathrm{Zar}}$ is the natural map. The main result of the paper is that the Bloch-Kato conjecture, i.e. the surjectivity of the norm residue homomorphism from Milnor K-theory to Galois cohomology $K_n^M(E)/m \to H^n(E, \mu_m^{\otimes n})$, is equivalent to the conjecture of Beilinson-Lichtenbaum. More precisely, assume that resolution of singularities holds over F and that the norm residue homomorphism in degree n is surjective for any extension E/F. Then the Beilinson-Lichtenbaum conjecture holds over F in weights at most n.

The authors start by reviewing the construction of motivic cohomology and the derived category of mixed motives [see V. Voevodsky, A. A. Suslin and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764197]. The category $DM^-(F)$ is the full subcategory of the derived category of bounded above complexes of Nisnevich sheaves with transfers and homotopy invariant cohomology sheaves. The motivic complex $\mathbf{Z}(n)$ is a specific object of this category, and motivic cohomology of a variety X over F is defined to be $H^i_M(X, \mathbf{Z}(n)) = \mathrm{Hom}_{DM^-(F)}(M(X), \mathbf{Z}(n))$, where $M(X)$ is an object of $DM^-(F)$ naturally associated to X. This is in fact isomorphic to the hypercohomology in the cdh-topology of $\mathbf{Z}(n)$ on X, and if X is smooth over F, then it suffices to take the Nisnevich topology.

Several properties of these motivic cohomology groups are given, for example a Mayer-Vietoris exact sequence for open covers, a projective bundle formula, and a blow-up exact sequence. The natural isomorphism of graded rings $\bigoplus_n K_n^M(E) \cong \bigoplus_n H_M^n(E, \mathbf{Z}(n))$ gives an interpretation of the Bloch-Kato conjecture in terms of motivic cohomology.

In the second half of the paper the actual proof takes place. The theorem is easily reduced to the case of a field, and only injectivity is hard. The main idea is to use the motivic cohomology of the boundary of the r-simplex $\partial\Delta^r$ (a singular scheme) to shift degrees. More precisely, if S is the affine line $\mathbf{A}_E^1$ with the points 0 and 1 identified to the point p, then the map $H_M^i(E, \mathbf{Z}/m(n)) \to H^i(E, \mu_m^{\otimes n})$ is a direct summand of the map

$$H_M^{n+1}(\partial\Delta_E^{n-i+1} \times S, \mathbf{Z}/m(n)) \to H^{n+1}(\partial\Delta_E^{n-i+1} \times S, \mu^{\otimes n}).$$

Every class coming from $H_M^i(E, \mathbf{Z}/m(n))$ vanishes in some neighborhood U of the vertices, hence comes from the motivic cohomology with supports $H_T^{n+1}(\partial\Delta_E^{n-i+1} \times S, \mathbf{Z}/m(n))$, for T equal to $\partial\Delta_E^{n-i+1} \times S - U$. Finally, by purity and induction, the map is injective on the latter group.

Thomas Geisser

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Voevodsky, Vladimir

Reduced power operations in motivic cohomology.

Publications Mathématiques. Institut de Hautes Études Scientifiques (2003), *no.* 98, 1–57.

MR2031199 (2005b:14038b) 14F42; 12G05, 19D45, 19E15

Voevodsky, Vladimir

Motivic cohomology with Z/2-coefficients.

Publications Mathématiques. Institut de Hautes Études Scientifiques (2003), *no.* 98, 59–104.

follows from CnoFile 2 031 198remarks The papers under review present Voevodsky's proof of the "Milnor conjecture", a remarkable achievement which marks the culmination of Voevodsky's program to extend Grothendieck's constructions of new "topologies", incorporate the philosophy of motives, and integrate into abstract algebraic geometry important techniques of homotopy theory. Voevodsky's work has inspired considerable further work by algebraic geometers and algebraic topologists, and holds great promise for dramatic new geometric results.

The fundamental theorem of Voevodsky states that if k is a field of characteristic different from 2 then the Galois cohomology groups $H^i(k, \mathbb{Z}/2)$ are generated by classes in $H^1(k, \mathbb{Z}/2)$. More precisely, J. Milnor [Invent. Math. **9** (1969/1970), 318–344; MR0260844] conjectured that the norm residue symbol determines an isomorphism

$$K_*^M(k) \otimes \mathbb{Z}/2 \ \tilde{\to} \longrightarrow H^*(k, \mathbb{Z}/2), \tag{1}$$

where $K_*^M(k)$ is the Milnor K-theory of the field k defined as the quotient of the tensor algebra on the multiplicative group k^* by the ideal generated by elements of the form $a \otimes b$ with $a, b \in k^*,\ a + b = 1$. Indeed, there is a conjectural generalization formulated by K. Kato [J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 3, 603–683; MR0603953] (the so-called Bloch-Kato conjecture) of this Milnor conjecture (1) applicable to any prime number l which asserts that the norm residue homomorphism determines an isomorphism

$$K_*^M(k) \otimes \mathbb{Z}/l \ \overset{\sim}{\to} \longrightarrow H^*(k, \mathbb{Z}/\mu_l^{\otimes *}) \tag{2}$$

for any prime l. Voevodsky has written these two papers so that whenever possible the results are proved for all primes. In a forthcoming paper with D. Orlov and A. Vishik ["An exact sequence for Milnor's K-theory with applications to quadratic forms", preprint, `arxiv.org/abs/math/0101023`], Voevodsky uses his proof of the Milnor conjecture to prove a companion conjecture of Milnor's [op. cit.] relating $K_*^M(k) \otimes \mathbb{Z}/2$ to the sections of the natural filtration of the Witt ring of quadratic forms over k.

The first of these papers verifies existence and important properties of cohomology operations in motivic cohomology. Earlier work of Voevodsky and joint work of Voevodsky, A. A. Suslin and E. M. Friedlander [*Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764197] established many good properties of motivic cohomology, and Voevodsky has shown [Int. Math. Res. Not. **2002**, no. 7, 351–355; MR1883180] that the motivic cohomology groups agree with the higher Chow groups of S. J. Bloch [Adv. in Math. **61** (1986), no. 3, 267–304; MR0852815]. The relationship of motivic cohomology to algebraic K-theory for smooth varieties over a field closely parallels the relationship of singular cohomology to (complex) topological K-theory of a topological space. Indeed, Voevodsky's results are results about motivic cohomology, and these results translate directly into results concerning algebraic K-theory with finite coefficients [see, for example, E. M. Friedlander and A. A. Suslin, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875; MR1949356].

The construction and essential properties of these cohomology operations are challenging to verify. The proofs are much more than a mere translation of corresponding results in algebraic topology. In the first paper, Voevodsky establishes results needed for the proof of the Milnor conjecture: construction of the Steenrod p-th power operations P^i, their relationship to the Bockstein operation, the Cartan formula, and the Adem relations. These are established in the context of the pointed motivic homotopy category $H_\bullet(k)$ considered by F. Morel and Voevodsky [Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 45–143 (2001); MR1813224]. Indeed, this extension of the category of k-varieties is essential for the formulation as well as proof of many results (e.g., Thom isomorphism and suspension isomorphism, as well as the representability of motivic cohomology by "Eilenberg-Mac Lane objects"). Other properties (uniqueness of P^i; the identification of the ring of all stable cohomology operations) not needed for Voevodsky's proof of the Milnor conjecture as presented in the second paper are not proved here.

The second paper provides Voevodsky's proof of the Milnor conjecture, referring freely to earlier papers by Voevodsky, M. Rost, and Suslin and Voevodsky for important subsidiary results as well as to the preceding paper on motivic cohomology operations. In some sense, one can view this paper as presenting the "master plan", with details to be found elsewhere. For example, no reference is given to the fact that Milnor K-group $K_M^n(k)\otimes\mathbb{Z}/l$ of a field k can be viewed as the (Zariski) mod-l motivic cohomology $H^n_{\mathrm{Zar}}(k,\mathbb{Z}/l(n))$ of k, and only a brief sketch is given of the fact that the Galois cohomology $H^n(k,\mu_l^{\otimes n})$ can be viewed as the (étale) mod-l motivic cohomology $H^n_{\text{ét}}(k,\mathbb{Z}/l(n))$ of k. The proof of the cohomological interpretation of $K_M^n(k)\otimes\mathbb{Z}/l$ was given by Bloch in [op. cit.] (in the context of his higher Chow groups); a proof of the second is outlined by Voevodsky with a reference to [V. Voevodsky, C. Mazza and C. Weibel, "Lectures on motivic cohomology. I", `math.rutgers.edu/~weibel/motiviclectures.html`] for a detailed proof. Voevodsky proves, as conjectured by A. A. Beilinson [in *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 1–25, Lecture Notes in Math., 1289, Springer, Berlin, 1987; MR0923131] and S. Lichtenbaum [in *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, 127–138, Lecture Notes in Math., 1068, Springer, Berlin, 1984; see MR 85i:11001 MR0756089], that the natural map determines an isomorphism

$$H^p_{\mathrm{Zar}}(k,\mathbb{Z}/l(q)) \ \tilde{\to}\longrightarrow H^p_{\text{ét}}(k,\mathbb{Z}/l(q)),\quad p\le q, \tag{3}$$

for $l=2$; in particular, he affirms the Milnor conjecture (1). As shown earlier by Suslin and Voevodsky [in *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, 117–189, Kluwer Acad. Publ., Dordrecht, 2000; MR1744945] and then extended by T. Geisser and M. N. Levine [J. Reine Angew. Math. **530** (2001), 55–103; MR1807268], one need only prove the surjectivity assertion of the Bloch-Kato conjecture (2) to conclude via an inductive argument the Beilinson-Lichtenbaum isomorphism (3).

Voevodsky's effort is dedicated to proving a higher-order version of the "Hilbert theorem 90". In Voevodsky's terminology, k satisfies $H90(n,l)$ if the l-adic étale cohomology group $H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(l)}(n))$ vanishes. For $n=1$, one can interpret this vanishing as a restatement of the classical Hilbert theorem 90, and for $n=2$ this is essentially the famous result of A. S. Merkur′ev and Suslin [Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136; MR0675529], which we may view as the confirmation of the Bloch-Kato conjecture in weight 2. One readily sees that $H90(n,l)$ implies the surjectivity assertion of Bloch-Kato in weight n. The main result of Voevodsky is that $H90(n,2)$ is valid for any field k and any $n\ge 0$. (For k of characteristic l, $H90(n,l)$ was established by Geisser and Levine [op. cit.].)

Voevodsky proceeds to prove $H90(n,2)$ by induction on n; thus, one begins by assuming the validity of $H90(n-1,2)$; in fact, he assumes $H90(n-1,l)$ for an arbitrary prime l and proceeds quite far towards the proof of the general Bloch-Kato conjecture before restricting to the case $l=2$. Voevodsky makes the observation that $H90(n-1,l)$ implies (2) for weights $q<n$ (and this choice of prime l) as well as a version of the Hilbert theorem 90 for K_q^M with $q<n$. Using "classical" techniques of Galois cohomology, Voevodsky then shows that these two conditions imply the vanishing of $H^n_{\text{ét}}(k,\mathbb{Z}/l)$ provided that k satisfies two conditions: (i) k has no extensions of degree prime to l; and (ii) $K_n^M(k)$ is l-divisible. Reasonably straightforward arguments reduce the required cohomological vanishing of $H90(n,l)$ to the vanishing of $H^n_{\text{ét}}(k,\mathbb{Z}/l)$ in this case, so that it remains to prove that we can pass from our given field k to a field extension K/k satisfying these two conditions as well as the injectivity

$$H^{n+1}_{\text{ét}}(k,\mathbb{Z}_{(l)}(n))\hookrightarrow H^{n+1}_{\text{ét}}(K,\mathbb{Z}_{(l)}(n)) \tag{4}$$

of the induced map. Condition (i) for a field extension K/k satisfying (4) can be easily arranged using a transfer argument. The heart of the proof is showing that Condition (ii) can also be arranged for K/k satisfying (4) when $l = 2$.

In order to arrange the 2-divisibility of $K_n^M(k)$, Voevodsky chooses a symbol $\underline{a} = (a_1, \ldots, a_n)$ representing a generator of $K_n^M(k)$ and takes K to be the function field of the associated norm quadric $Q_{\underline{a}}$. It is well known that the class in $K_n^M(k)$ associated to $\underline{a}$ is divisible by l. The challenge is to prove (3) for K/k. Up to this point, the proof has been largely inspired by the proof of Merkur′ev and Suslin for the Bloch-Kato conjecture in weight 2 [op. cit.]. Voevodsky proceeds to investigate the motivic cohomology of the Čech simplicial scheme $\mathcal{X}_{\underline{a}}$ associated to the norm quadric $Q_{\underline{a}}$. He employs his motivic cohomology operations and the vanishing of "Margolis homology" of a closely related simplicial scheme to prove that $H90(n-1, 2)$ implies that $H^{n+1}(\mathcal{X}_{\underline{a}}, Z_{(2)}) = 0$. Now Voevodsky invokes results of Rost ["On the spinor norm and $A_0(X, K_1)$ for quadrics", preprint, 1988, `www.mathematik.uni-bielefeld.de/~rost/spinor.html`; "Some new results on the Chow groups of quadrics", preprint, 1990, `www.mathematik.uni-bielefeld.de/~rost/chowqudr.html`; J. Ramanujan Math. Soc. **14** (1999), no. 1, 55–63; MR1700870] concerning the motive of the norm quadric $Q_{\underline{a}}$ to obtain the necessary injectivity by relating the motive of $\mathcal{X}_{\underline{a}}$ to that of the field k.

Eric M. Friedlander

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MR2811603 (2012j:14030) 14F42; 19D45

Voevodsky, Vladimir

On motivic cohomology with Z/*l*-coefficients.

Annals of Mathematics. Second Series **174** (2011), *no.* 1, 401–438.

This landmark paper completes the publication of Voevodsky's celebrated proof of the Bloch-Kato conjecture—it is now a theorem that the norm residue homomorphism

$$K_n^M(k)/l \to H^n_{\text{ét}}(k, \mu_l^{\otimes n})$$

is an isomorphism for all fields k, all primes l with $(l, \operatorname{char} k) = 1$ and all n. The norm residue homomorphism is a special case of a comparison morphism between (Beilinson) motivic cohomology and (Lichtenbaum) étale motivic cohomology, and indeed the more general Beilinson-Lichtenbaum conjecture is also a consequence of the results of the paper. The proof of these conjectures is one of the fundamental results in algebraic K-theory and motivic cohomology, and has involved a lot of time as well as a lot of work by a lot of people. The result itself is a great piece of mathematics that allows a much better understanding of the relation between motivic cohomology or algebraic K-theory with their étale counterparts, but even more importantly, the methods developed to prove it (derived categories of motives, motivic cohomology and homotopy) have already had a large impact on mathematics and will continue to do so in the years to come. Needless to say, Voevodsky was awarded the Fields Medal in 2002 for developing these methods leading to a proof of the case $l = 2$, which had been known as Milnor's conjecture [cf. Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 59–104; MR2031199].

An introduction to the proof of the Milnor conjecture can be found in [F. Morel, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 2, 123–143; MR1600334], while an introduction to the proof of the Bloch-Kato conjecture can be found in [C. A. Weibel, in *Some recent developments in algebraic K-theory*, 277–305, ICTP Lect. Notes, 23, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008; MR2509183].

The proof for both $l = 2$ and l odd combines homotopy-theoretic techniques and an algebraic-geometric study of norm varieties associated to symbols. On the homotopy-theoretic side one has Voevodsky's construction of derived categories of motives [cf. V. Voevodsky, A. A. Suslin and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764197] and the Morel-Voevodsky $\mathbf{A}^1$-homotopy theory [F. Morel and V. Voevodsky, Inst. Hautes Études Sci. Publ. Math. No. 90 (1999), 45–143 (2001); MR1813224]. These are used to define motivic cohomology and study cohomological operations in motivic cohomology with finite coefficients via motivic Eilenberg-Mac Lane spaces. On the geometric side, the norm varieties and their associated (generalized) Rost motives provide the nontrivial input for the cohomological operations machine to work. There are some differences between the cases $l = 2$ and l odd. On the homotopy theory side, some additional results on cohomology operations in motivic cohomology are necessary in the odd prime case. On the geometric side, the case $l = 2$ uses Pfister quadrics and their associated Rost motives. For l odd, these have to be replaced by ν_n-varieties, resp. $\nu_{\leq n}$-varieties, and generalized Rost motives.

The strategy for the Bloch-Kato conjecture was already described in Voevodsky's work on the Milnor conjecture: one first reduces to fields which have characteristic 0 to be able to use results on motivic Eilenberg-Mac Lane spaces [Publ. Math. Inst. Hautes Études Sci. No. 112 (2010), 1–99; MR2737977] and motivic duality theorems [in *Cycles, transfers, and motivic homology theories*, 188–238, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000; MR1764202]. The comparison between motivic cohomology in the Nisnevich and the étale topology then follows from a vanishing $H^{n+1}_{\text{ét}}(k, \mathbf{Z}_{(l)}(n)) = 0$. To inductively prove these higher versions of Hilbert's theorem 90, it suffices to find for each symbol $a \in K^M_n(k)/l$ a field K such that

1. a vanishes in $K^M_n(K)/l$ and
2. the morphism $H^{n+1}_{\text{ét}}(k, \mathbf{Z}_{(l)}(n)) \to H^{n+1}_{\text{ét}}(K, \mathbf{Z}_{(l)}(n))$ is an injection.

The injectivity is proved in the paper under review, using results on the existence and properties of $\nu_{\leq n}$-varieties which split symbols in Milnor K-theory. The results on $\nu_{\leq n}$-varieties were announced by Rost and proved in [A. A. Suslin and S. Joukhovitski, J. Pure Appl. Algebra **206** (2006), no. 1-2, 245–276; MR2220090].

We now give a more detailed overview of the arguments in the paper which prove this injectivity. In section 2, an analysis of the structure of motivic Eilenberg-Mac Lane spaces and computations in motivic cohomology are used to show a uniqueness theorem for cohomology operations

$$\widetilde{H}^{2n+1}(-, \mathbf{Z}/l(n)) \to \widetilde{H}^{2nl+2}(-, \mathbf{Z}/l(nl)).$$

In section 3, symmetric powers of relative Tate motives are used to construct explicitly one particular such operation, which by the previous uniqueness agrees with the composition βP^n of the Bockstein and the motivic reduced power operation. Section 4 then establishes the vanishing of Margolis homology for embedded simplicial schemes related to ν_n-varieties as well as a generalization of the motivic degree theorem.

Section 5 contains one of the major points—the construction and properties of generalized Rost motives. This is the replacement for the Rost motives associated to Pfister quadrics in the case $l = 2$. For an embedded simplicial scheme $\mathcal{X}$ and a cohomology class $\delta \in H^{n+1}(\mathcal{X}, \mathbf{Z}/l(n))$ with $Q_0 Q_1 \cdots Q_n(\delta) \neq 0$, Voevodsky constructs a (self-dual) generalized Rost motive M_{l-1} over $\mathcal{X}$ as the $(l-1)$@-st symmetric power of the cone of the class δ. If additionally there exists a ν_n-variety X whose motive $M(X)$ lies in $DM_{\mathcal{X}}$, the results of section 4 show that this generalized Rost motive M_{l-1} is pure and splits off as a direct summand of $M(X)$.

Finally, section 6 uses the results of the previous sections to establish the injectivity assertion (2). Voevodsky considers the Čech simplicial scheme $\mathcal{X} = \check{C}(Y)$ of the disjoint union Y of all smooth schemes which split the symbol a. This simplicial scheme satisfies the conditions of Section 5, so there is a generalized Rost motive associated to it and the injectivity on étale motivic cohomology reduces to show the vanishing of $H^{n+1}(\mathcal{X}, \mathbf{Z}_{(l)}(n))$. The vanishing of Margolis homology from section 4 allows one to use Milnor operations to embed this group into $H^{2lb+2}(\mathcal{X}, \mathbf{Z}_{(l)}(lb+1))$, whose vanishing follows from properties of the splitting $\nu_{\leq(n-1)}$-varieties.

Matthias Wendt

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The Univalent Foundations Program

Homotopy type theory—univalent foundations of mathematics. (English)

The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ,, 2013, xiv+589 pp.

This is an important book, and it is the first one on a research topic called Homotopy Type Theory (HoTT). This new theory mixes two apparently distant disciplines: type theory (a branch of logic) and homotopy theory (a branch of algebraic topology). The topic originated from Voevodsky's contributions, which were preceded by the work of several authors (among them, Hofmann, Streicher, Awodey and Warren). Before describing the contents of the book, some meta-information deserves comment.

First, the book is a collaborative effort, fruit of a special year (2012–13) at the Princeton Institute for Advanced Study. The book is published on an open access basis, and the authorship is collective (there is a long list of contributors, but no name is particularly highlighted). All this is quite original in our days.

Second, the writing of the text has the explicit objective of finding an *informal* style of expressing formal type theory, emulating the way working mathematicians write their papers without paying attention to foundations. However, the (unnamed) authors always try to keep a link with the real formal development; in fact, it is stated in the book that most of the results presented were first found in a formalized setting (on occasions using the Coq or Agda proof assistants) and then "unformalized". This is a very promising point of view, but it is not without difficulties (some examples will be commented on later).

The contents of the book are separated into two parts. Chapters 1 to 7 deal with basics, while Chapters 8 to 11 are devoted to formalizing concrete mathematics using the language of HoTT. The book ends with an appendix which describes the particular type theory where the results of the book could be expressed.

After a quite long and very enlightening introduction, Chapter 1 introduces dependent type theory, and Chapter 2 presents the actual topic. It consists of a new interpretation of identity types, which were introduced by Martin-Löf in his intensional dependent type theory. The main idea is to interpret a proof of equality between two terms a, b of type A as a *path* from a *point* a to a point b in a (topological) *space* A. Then, the formal properties of identity (reflexive, symmetric, transitive) can be read (almost) as defining a groupoid structure over the "space" (type) A. This is *almost* a groupoid because the proofs of the equations are again terms of a type. If we denote by $\mathrm{Id}_A(\_,\_)$ the identity type over a type A, from $p : \mathrm{Id}_A(a,b)$, a proof $p^{-1} : \mathrm{Id}_A(b,a)$ can be constructed, but the *equation* $p^{-1} \circ p = 1$ is to be understood as the construction of a term $q : \mathrm{Id}_{\mathrm{Id}_A(a,a)}(p^{-1} \circ p, \mathrm{refl}_a)$, where $\mathrm{refl}_a : \mathrm{Id}_A(a,a)$ is the primitive term expressing the reflexive property of equality. Or, through the homotopy glasses, this means that $p^{-1} \circ p$ is homotopic to the constant path refl_a. Such complex proofs as q devised on $\mathrm{Id}_A(\_,\_)$ show equality as having a meaning beyond a simple boolean operator (we are in the presence of *proof-relevant* mathematics); indeed, $\mathrm{Id}_A(\_,\_)$, being also a type, carries a groupoid-like structure, too. In that way, any type is endowed with an ∞-groupoid structure (where equations are established up to homotopy). From Grothendieck's claim that ∞-groupoids encapsulate *homotopy types*, we get an unexpected link between types as occurring in type theory and homotopy types from homotopy theory.

Chapter 2 also explains that taking full advantage of these ideas requires another tool: Voevodsky's univalence axiom. It is stated in a context where a universe $\mathcal{U}$ (a type such that its terms are also types) is fixed. A definition is given of when two types $A, B : \mathcal{U}$ are equivalent $\mathrm{Equi}_{\mathcal{U}}(A,B)$. Even if the concept of equivalence can be defined in several different ways (Chapter 4), we can think of it as meaning that there is an arrow $f A \to B$ such that it is known there is a $g B \to A$ satisfying $f \circ g = 1$ and $g \circ f = 1$. But the equality $g \circ f = 1$ means really that a term $p : \mathrm{Id}_{A \to A}(g \circ f, \mathrm{id})$ is constructed; that is to say, $g \circ f$ is homotopic to the identity map from A to A; so the terms inhabiting $\mathrm{Equi}_{\mathcal{U}}(A,B)$ can be reasonably interpreted as homotopy equivalences between the *spaces* A and B. There is a natural map going from $\mathrm{Id}_{\mathcal{U}}(A,B)$ to $\mathrm{Equi}_{\mathcal{U}}(A,B)$. The univalence axiom imposes that this arrow is an equivalence for all $A, B : \mathcal{U}$. Therefore the univalence axiom "invents" an arrow $ua\, \mathrm{Equi}_{\mathcal{U}}(A,B) \to \mathrm{Id}_{\mathcal{U}}(A,B)$, transforming equivalences into identities.

Univalence is used in Chapter 3 to study a stratification of types, depending on the richness of their identity types. For instance, a type is considered a *set* if there is at most one equality proof for each pair of its terms. The Axiom of Univalence proves that there are types which are not sets (in particular, this is the case of the fixed universe $\mathcal{U}$). After studying equivalences in Chapter 4, Chapter 5 is devoted to inductive definitions. Here, the univalence axiom is used, for instance, to prove the uniqueness of the object of *natural numbers* $\mathbb{N}$ in HoTT.

In order to empower the homotopic interpretation another ingredient is however needed: higher inductive types (Chapter 6). The usual inductive types (Chapter 5) define a new type by freely creating new terms through constants and constructors. In higher inductive types, in addition, some identities (terms of identity types) are also created. It amounts, in a sense, to defining a collection of terms and, simultaneously, a quotient over that collection. Higher inductive types allow the introduction of spaces by attaching cells (such as the spheres S^n, for instance) and the concept of n-type (Chapter 7).

The second part of the book (formalization of mathematics) starts in Chapter 8, where further relations between HoTT and homotopy theory are explored. Chapter 8 begins with a well-known result from algebraic topology which is a groundbreaking theorem in HoTT: $\pi_1(S^1) = \mathbb{Z}$, the fundamental group of the circle S^1 is isomorphic to the integers. In this proof not only is the univalence axiom required but also higher inductive types (used, in particular, to define both S^1 and the homotopy groups). Other important results in algebraic topology are also translated to HoTT in this chapter.

Category theory is considered in Chapter 9, studying different possible definitions in HoTT as well as the implications of these alternative concepts. Chapter 10 concentrates on the category of *sets*, discussing the relations with previous results in topos theory. The last chapter of the book (Chapter 11) deals with analysis and the reals. It is interesting in its development and also acts as an illustration for the fact that HoTT can be a language for other parts of mathematics far from logic, homotopy or categories.

The non-expert will read the book with a mixture of pleasure and surprise, seeing how, from a rather sober formal language, some difficult results (in algebraic topology, in particular) are replayed in an original way.

As can be expected in so young a discipline, light comes with some shadows. These are related to language and intuition (and the strains between them), to the completeness of the development and, finally, to the foundational character of the theory.

A confusing use of language appears when it is claimed that HoTT allows the *computation* of an object, as in the case, for example, of homotopy groups. It is one thing to *prove* that a homotopy group is isomorphic to a concrete group (a concrete group that must be previously known by other means) and another thing is to describe an automated procedure to determine such a group. The distinction is important to make precise what one can expect about HoTT (in its current degree of development). Only at the end of Chapter 8 is the theorem stated that, for all $n \geq 3$, $\pi_{n+1}(S^n) = \mathbb{Z}_k$ for some k, which could raise the issue of a real *computation* to determine k (but, even so, all the development would be directed by the fact that $k = 2$ would be the expected result).

The trade-off between informal presentation and rigorous formal proofs makes some parts of the text difficult to follow. For instance, the word *mere* (and its variants *merely*, etc.) refers to a very concrete technical constraint, and then some statements and descriptions of proofs look rather contorted. Perhaps some more formal notation could be of help in those cases.

Commenting now on the formal language itself, since the concrete type theory in which HoTT is being described is, at the same time, being introduced, some doubts arise about whether some definitions are forced to respond to the intuitions the authors had in mind. This for example happens when discussing the types A where there exists a term a such that $\mathrm{Id}_A(a, b)$ is inhabited for any other term $b : A$. After insisting that identities should be interpreted as paths, one could guess that such a type A would be identified with a (path-)connected space. In contrast, such a space is defined as *contractible*; the reason given ("the meaning of 'there exists' in this sentence is a continuous/natural one") does not seem very convincing. Another example can be found on page 175: "The same is true in type theory, *if we formulate these conditions appropriately*" (emphasis by the reviewer). This point seems especially frail in the decisions involving higher inductive types, where to impose equalities *judgmental* or *propositional* could have important consequences in the resulting theory.

With respect to the completeness of the approach, the difficulties are not hidden and they are commented on in detail throughout the book. Nevertheless, it is not until the end of the book (the last paragraphs of the last section of the appendix) that some consequences of the introduction of the univalence axiom are presented. For instance, univalence implies that there is, at least up to now, no proof of the *canonicity* of the type theory (that is to say, there could be a term of type $\mathbb{N}$ that does not reduce to a numeral) and no proof of the constructive nature of the theory (in particular, it is not known whether it could be enhanced with a *computational content*). As for the consistency of the theory, a simplicial model is known (due to Voevodsky), ensuring consistency up to that of Zermelo-Fraenkel with the Axiom of Choice. It is a highly non-constructive model and, furthermore, covers the Axiom of Univalence but not higher inductive types.

From the completeness point of view, the concept of higher inductive type is the weakest aspect of the book. These types are imperative (in conjunction with the univalence axiom) for the most appealing results of the theory, but nevertheless there is not a complete treatment of them (even from the syntax point of view). This could be justified if their interpretation could be considered straightforward, but this does not seem to be the case. In fact, in the examples developed, the mentioned decisions about a propositional equality imply the occurrence of numerous axioms (it is laboriously manifest when trying to emulate the corresponding induction schemes with the proof assistant Coq), which could, in principle, raise some doubts about the consistency of the theory, depending on the higher inductive definitions allowed.

These gaps make the choice of the subtitle of the book ("univalent foundations of mathematics") debatable, although it is true that HoTT is presented frequently in other publications as a new foundation for mathematics. The incomplete specification of higher inductive types, the lack of a constructive semantics and the excessive reliance on types and terms whose existence is imposed by axiom, and not by construction, make it premature to claim that new foundations of mathematics are described in the book. Several current lines of research are intended to overcome these shortcomings.

These observations do not cloud at all either the interest of the theory or, of course, the importance of the book under review. It will last as an accessible and rigorous introduction to a new and fascinating discipline under way. As it is being developed by an extremely competent team of researchers, in a collective endeavor without any precedent, undoubtedly it will continue producing enlightenments in the future.

Julio Rubio

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Voevodsky, Vladimir

Products of families of types and (Π, λ)-structures on C-systems.

Theory and Applications of Categories **31** (2016), Paper No. 36, 1044–1094.

In his 1978 Ph.D. thesis [*Generalised algebraic theories and contextual categories*, Oxford Univ., 1978], John Cartmell introduced the notion of *contextual category* for the purpose of formulating the categorical semantics of dependent type theories as introduced by Per Martin-Löf around 1970. This notion was used a decade later in the reviewer's thesis [*Correctness and completeness of a categorical semantics of the calculus of constructions*, Univ. Passau, 1988], which studied the categorical semantics of T. Coquand and G. P. Huet's calculus of constructions.

In a series of papers the author has started a very detailed investigation of contextual categories, now called C-systems, in a form which is as detailed and formalist as required for a blueprint for a formalization in a computer-assisted system such as Coq.

The main mathematical contribution of the paper under review is a reformulation of Cartmell's account of dependent types (here called "Cartmell-Streicher structures") as so-called (Π, λ)-structures. This reformulation is closer to syntax than previous formulations which are closer in spirit to the "quantifiers as adjoints" paradigm introduced by F. W. Lawvere in the late 1960s. In this sense there is some similarity with P. Dybjer's notion of "categories with families" which, however, is even further away from the spirit of categorical logic.

In the reviewer's opinion, more categorical accounts, based on Grothendieck fibrations and formulated in terms of "display" maps, are more transparent and more concise. Of course, in order to interpret syntax, these more abstract versions have to be endowed with a so-called "splitting" (in the sense of "split fibrations" as already considered by A. Grothendieck and his school). Such split models can be transformed (purely mechanically) into models which are closer to syntax like the C-systems favored by the author.

The style of exposition is dictated by the author's desire to give a blueprint for formalization. This goes as far as renaming "Theorems" as "Problems" and "Proofs" as "Constructions", which certainly is in accordance with the Curry-Howard paradigm of "propositions-as-types".

This paper clearly demonstrates how mathematics will change when it is formulated in a style ready-made for formalization in computer-based proof assistants. Certainly, different readers will come to different conclusions whether such a change is beneficial or not. I suspect that most mathematicians will stay at least skeptical.

Thomas Streicher

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