Handbook of enumerative combinatorics, Miklós Bóna (Editor), Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2015, xxiii+1061 pp., ISBN 978-1-4822-2085-8, hardcover

The *Handbook of enumerative combinatorics* celebrates a coming of age for the field of combinatorics. Most other branches of combinatorics overlap with enumeration. This includes algebraic combinatorics, analytic combinatorics, graph theory, combinatorial geometry, as well as newer topics, such as additive combinatorics, graphons, and cluster algebras. None of these can be done without an understanding of enumeration. Conversely, as reflected by the *Handbook*, any comprehensive work on enumeration will touch most of these areas.

At the time of publication of Berge's *Principles of Combinatorics* [Ber71], combinatorics was known as "little more than a bag of tricks". These were the words of Richard Stanley, whose seminal two-volume work, published over the period 1986–1991, marks more or less the beginning of the evolution away from the bag of tricks epithet.

How can one tell when a field ceases to be a bag of tricks? One answer is that structures will arise whose study becomes a fundamental pursuit in itself, typically using techniques from all classical branches of mathematics (analysis, algebra, geometry, topology) and sometimes yielding results of independent interest in these other branches. One might remark on the incidence algebra of a partially ordered set [Sta97], the structure theory of recurrences [GJ83, Joy81], the analytic study of generating functions [FO90, Wil94], or the topological study of complex hyperplane arrangements [OT92].

The *Handbook*'s goal is to present combinatorial methods in their essence, not to review adjacent areas of mathematics with which combinatorial theory intertwines. This is necessary, unless the goal of keeping it to one volume is abandoned in favor of a Bourbaki-style approach, spiraling into many thousands of pages. The purpose of this review is to provide the complementary narrative of unification with other areas of mathematics. By this means, readers of the *Handbook* might become aware, for example, of the relations between transfer matrices and Pfaffian determinants as they occur in Section 1.4 (linear algebraic methods), Chapter 8 (words), Chapter 9 (tilings), and Chapter 10 (lattice paths), or their applicability to counting problems in Section 12.2 (growth rates of permutation classes) and some of the problems in Chapter 2.

The earliest and most obvious connections to existing mathematics of considerable depth show up in algebraic combinatorics. MacDonald's book on symmetric function theory [Mac79], reviewed in the *Bulletin* in 1981 [Sta81], was the culmination of decades of work understanding the structure of subalgebra in $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials symmetric under the natural action of $\mathcal{S}_n$. The single most studied combinatorial objects are the *Young tableaux*. These arise from representation theory. Their shapes (Young diagrams) index the irreducible representations of the symmetric group. Filling in these shapes with integers, while following various rules,

computes representation-theoretic invariants. Generating functions that enumerate these filled-in shapes (Young tableaux) turn out to lead to unforeseen bijections, shedding light on the representations from whence they came. These structures are featured in Chapters 11 and 14 of the *Handbook of Enumerative Combinatorics*. Although their basic theory has been understood for 50 years, the associated bivariate generating functions (Chapter 11) and non-classical shapes (Chapter 14) are quite recent.

Turning to geometric combinatorics, let us consider *hyperplane arrangements*: finite collections of hyperplanes in real or complex euclidean space. Questions about these arose originally in the classification of finite groups. Specifically, associated with each Coxeter group is a set of hyperplanes of reflection; to understand the group one must understand the lattice of intersections of these hyperplanes and the geometry of the chambers in the complement of the arrangement in $\mathbb{R}^n$. The lattice may be completely described by linear algebra. The abstract properties of such lattices gives rise to the theory of matroids; see, e.g., [Wel76, Dil78]. To understand the chambers, it is fruitful to complexify. In the 1970s and 1980s, substantial progress was made in understanding the topology of the complement of a hyperplane arrangement in $\mathbb{C}^n$. Section 1.7 of the *Handbook* surveys many of these results. The characteristic polynomial for an arrangement is the generating function for the lattice of intersections, with each flat weighted by its Möbius function $\mu(0, F)$. The Poincaré polynomial for a topological space is the generating function for the ranks of its homology groups. Section 1.7 begins with Zazlavsky's 1975 result [Zaz75] using the characteristic polynomial to count regions of the real complement and the relation between this and the Poincaré polynomial of the complexification discovered in 1980 by Orlik and Solomon [OS80]. Sagan's 1999 result on the factorization of the characteristic polynomial relies on the 1980 results of Terao [Ter80] and Sato [Sat80] concerning the structure of the subalgebra of derivations dividing $D(\alpha_H)$ for all planes H in the arrangement. A more complete understanding of these results involves stratified Morse theory [GM88]. Fifty pages of the *Handbook* are devoted to hyperplane arrangements and the theory of matroids.

Generating functions are fundamental to all branches of combinatorics. A power series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ encodes an array of real or complex numbers $a_{\mathbf{r}}$ indexed by $\mathbb{Z}^d$ (here $\mathbf{z}^{\mathbf{r}}$ denotes the monomial $z_1^{r_1} \cdots z_d^{r_d}$). The relation between the analytic properties of F and properties of the numbers $\{a_{\mathbf{r}}\}$ forge a deep connection between combinatorics and complex analysis. Identities satisfied by the coefficients $\{a_{\mathbf{r}}\}$, such as convolutions, recursions, and substitutions, correspond to familiar algebraic equations for the generating functions. The generating function by definition encodes exact information about $a_{\mathbf{r}}$. This may be recovered via Cauchy's integral formula, $a_{\mathbf{r}} = (2\pi i)^{-d} \int \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) d\mathbf{z}/\mathbf{z}$. Approximate information about $a_{\mathbf{r}}$ follows from approximation of this integral, which is sometimes easier to extract and more useful. Chapter 2 of the *Handbook* details this process for univariate generating functions. The chapter is explicit about its great debt to the seminal 2009 work of Flajolet and Sedgewick [FS09]. Perhaps the most useful results in the chapter are the transfer theorems of Flajolet and Odlyzko [FO90]. Summarized in Section 2.17, these give conditions under which the asymptotic behavior of a function f near its minimum modulus singularity R, transfer to an asymptotic series development for

its coefficients $\{a_n\}$. This technique as well as others, such as Lagrange inversion (Section 2.6), the kernel method (Section 2.8), and the quasi-power method of Bender, Canfield, Gao, and Hwang (Section 2.22), are based on elementary complex analysis. Analogous results for multivariate generating functions require much stronger analytic and topological machinery. Write the multivariate Cauchy integral as $\int_{\mathcal{C}} \omega$, where ω is the holomorphic d-form $\mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \, d\mathbf{z}/\mathbf{z}$ and $\mathcal{C}$ is the homology class of a small torus in the complement of the variety $\mathcal{M} := \{F(\mathbf{z})\mathbf{z}^{\mathbf{1}} = 0\}$. An unexpected connection to the previous topic is that stratified Morse theory pops up again to give a decomposition of $\mathcal{C}$ into cycles in $H_d(\mathcal{M})$ over which ω can be integrated using multivariate residues and asymptotic techniques pioneered in [ABG70]. The multivariate story goes beyond what the *Handbook* can address, and it is the topic of the book [PW13].

Much can be determined about a generating function from the locations of its zeros, in particular from where the zeros are not.[1] Chapter 7 of the *Handbook* tells some of this story. Generating polynomials, whose roots are all real and negative, have coefficient sequences that are log-concave, hence unimodal. This observation goes all the way back to Newton. To prove the real root property, it is helpful to understand what operators on polynomials that preserve this property. A complete classification was obtained long ago by Pólya and Schur [PS14]. Flashing forward almost a hundred years, a remarkable multivariate version of this result was discovered by Borcea and Brändén [BB09a, BB09b]. To formulate this they had to find the correct multivariate generalization of the real root property. Building on prior work of Wagner, Sokal, and others, they define $F : \mathbb{C}^d \to \mathbb{C}$ to be *stable* if F cannot vanish when every coordinate takes a value in the open upper half-plane. This property might seem unwieldy; however, multivariate Pólya–Schur theory allows us to identify many classes of stable polynomials. The consequences of stability are significantly stronger than Newton's inequalities. For example, when the coefficients are interpretable as a probability distribution on $\mathbb{Z}^d$, the joint distribution is negatively associated. It turns out that the theory of stability simplifies in useful ways when restricted to multi-affine polynomials, namely those in which the maximum degree in each variable separately is 1. The resulting theory of *strong Rayleigh distributions*, set forth in [BBL09, 2009], solved a number of conjectures in probability theory and algebraic combinatorics, for example in 2011 to obtain better approximations in the Traveling Salesman Problem [GSS11].

An important tool for Brändén et al. in developing the theory of stable polynomials was *hyperbolicity*. This notion arose in Gårding's 1951 study of the stability of partial differential equations [Går51] and was exploited by Atiyah, Bott, and Gårding in their study of caustic integrals [ABG70]. This theory and its extensions by Brändén et al. underlie a number of very recent developments. The most notable is the 2014 proof of the Kadison–Singer conjecture [MSS15], which can be formulated as a spectral inequality for matrices or as a seemingly nonquantitative extension result for C^*-algebras. The Monotone Column Permanent Conjecture was proved using stability theory in 2011 [BHVW09]. Gurvitz used stability theory to prove a number of results, implying among other things a very short proof of van der Waerden's conjecture. While the result was already proved in 1979 [Ego81, Fal81],

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[1] Gian-Carlo Rota, the academic grandfather of the editor of the *Handbook*, once famously said, "The one contribution of mine that I hope will be remembered has consisted in pointing out that all sorts of problems of combinatorics can be viewed as problems of the location of the zeros of certain polynomials."

the new results by Gurvitz have promising implications for the computation of permanents. Hyperbolicity has also been exploited in convex programming, in the construction of self-concordant barriers [Gül97]. More details on hyperbolicity and its applications to discrete mathematics can be found in [Pem12].

Two developments not discussed in the *Handbook*, either because they are quite recent or because they touch less centrally on enumeration, are cluster algebras and additive combinatorics. Additive combinatorics studies structures and extremal properties of sets, such as the set $A + B := \{x + y : x \in A, y \in B\}$. The recent monograph [TV10] shows how these are studied by diverse methods, including those of probabilistic graph theory and harmonic analysis as well as functional analysis and ergodic theory. This area is young and rapidly evolving. For the second topic, a jumping-off point is the enumeration of tilings, a topic comprising Chapter 9 of the *Handbook*. Many tiling models arose first in the statistical physics literature. Chapter 9 discusses a number of methods of enumerating tilings, beginning with the Conway–Thurstan height function paradigm, and going on to include the transfer matrix method, Pfaffian determinants, spanning tree bijections, and the use of representation theory and gauge transformations. Of particular interest are abelian transformation methods described in Section 9.5, which include domino shuffling, urban renewal, and Kuo condensation. These involve transformations on weighted graphs associated with the tiling, which might be thought of as a kind of generalized series-parallel reduction. Keeping track of the weights, one finds that the algebraic relations between them, which are compositions of certain rational substitutions, rather surprisingly preserve the class of Laurent polynomials. In 2002, the algebra behind this phenomenon was uncovered [FZ02a, FZ02b]. The abstract formulation is known as the theory of cluster algebras.

The many parts of the *Handbook* I have not had a chance to touch on include chapters on the classical structures of discrete mathematics and theoretical computer science: graphs (Chapter 6), words (Chapter 8), trees (Chapter 4), lattice paths (Chapter 10), and planar maps (Chapter 5). The end of the chapter of planar maps hints at a number of interesting new developments, including the Brownian planar map, a continuum scaling limit of uniformly random planar quadrangulations [Mie13]. As pointed out in the chapter, "The topic is currently evolving at a very fast pace." Graph theory is a huge subject, the most interesting recent developments in which are somewhat outside the scope of enumeration. For example Lovasz's 2012 book on graphons [Lov12] is the *Bulletin*'s most recently reviewed combinatorics book [Neš14]. The *Handbook* contains a chapter on permutations with forbidden patterns (Chapter 12). While the jury is still out concerning the long-term ends of this research, the topic has generated quite a bit of recent research. This is documented by the 168 citations in Chapter 12, the vast majority of which are less than 20 years old. The last chapter, on computer algebra, is delightful. Computational ring theory and group theory are part of the new essential toolkit of any mathematician. At first glance it is not obvious that this chapter concerns enumeration (expect for the last part on on symbolic summation). The key is that when working with generating functions and formal power series, one constantly runs into the need for exact algebraic computation.

If I have any quibbles, they are not with the topics omitted—I hope I have made it clear that the encyclopedic approach necessitates leaving the more lyrical narrative to someone else—but with the organization. The chapter numbering is strange. The first two chapters comprise one quarter of the book and are packaged

together as "I: Methods", with the remaining thirteen falling under "II: Topics". But I and II are not Part 1 and Part 2, which refer instead to the first and second half of Chapter 1 (algebraic methods and discrete geometric methods, respectively), while Chapter 2 (analytic methods) is on its own. There is a logic to it; however, it might have been better to divide Chapter 1 into eight chapters, which along with Chapter 2 could then have been called Part I. This would also eliminate the four-deep section headers in Sections 1.3, 1.4, 1.5, 1.6, and 1.7. This quibble not withstanding, the *Handbook* is as advertised, deserving of a prominent place on your bookshelf.

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