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Metric geometry of locally compact groups, by Yves Cornulier and Pierre de la Harpe, Tracts in Mathematics, Vol. 25, European Mathematical Society (EMS), Zürich, 2016, viii+235 pp., ISBN 978-3-03719-166-8

The central topic of group theory is the study of symmetries of geometric and algebraic structures. The structure is reflected then in the algebraic properties of the group of symmetries. Intimate relation between group theory and geometry is, for example, the main idea of Felix Klein's Erlangen program. Another example of this relation is the study of discrete groups of isometries of the hyperbolic plane started by Felix Klein and Henri Poincaré, where the structure of the group is naturally described by the associated tessellations of the plane.

A straightforward way of transforming a finitely generated group into a geometric object is via the notion of the *Cayley graph*, introduced by Arthur Cayley for finite groups [Cay78]. Later, Max Dehn used them (under the name *Gruppenbild*) in the study of algorithmic problems in fundamental groups [Deh87]. The main idea of M. Dehn was to use the geometry of the space to understand algebraic structure of the group, i.e., to "geometrize" group theory.

If G is a group with a chosen finite generating set S, then its (right) Cayley graph is the graph with the set of vertices G, where two vertices $g_1, g_2 \in G$ are connected by an edge if and only if $g_1 = g_2 s$ for some $s \in S \cup S^{-1}$. The Cayley graph is a natural visualization of the group. For example, it is natural to imagine the infinite cyclic group $\mathbb{Z}$ as an infinite chain, and the free abelian groups $\mathbb{Z}^n$ as n-dimensional grids. These geometric visualizations are precisely the Cayley graphs.

The set of vertices of the Cayley graph is naturally a metric space: the distance between two vertices is the smallest number of edges in a path connecting them. This is the *word metric* on the group: the distance between g_1 and g_2 is equal to the shortest length of a representation of $g_1^{-1} g_2$ as a product of the elements of $S \cup S^{-1}$. This metric is *left-invariant* the distance between g_1 and g_2 is equal to the distance between hg_1 and hg_2 for all $g_1, g_2, h \in G$.

The idea of considering groups as metric spaces is implicitly present in the works of Max Dehn, and it was developed and popularized by Mikhael Gromov [Gro87, Gro93]. Geometric intuition, language, and methods have opened new approaches and directions in group theory: theory of hyperbolic groups [Gro87] (i.e., groups that are *negatively curved* as metric spaces) and its generalizations (e.g., various versions of the notion of nonpositively curved groups [BH99], relatively hyperbolic groups [Far98], acylindrically hyperbolic groups [Osi16], etc.); growth of groups [Gro81, Man12] and other asymptotic invariants of geometric nature [Gro93]. Modern group theory cannot be imagined without geometric methods.

The word metric on a finitely generated group G depends on the choice of the generating set. On the other hand, if d_1 and d_2 are the word metrics on the same group defined using two generating sets, then there exists $L > 1$ such that $L^{-1} d_1(g_1, g_2) \leq d_2(g_1, g_2) \leq L d_1(g_1, g_2)$ for all $g_1, g_2 \in G$. In other words, two

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word metrics are *bi-Lipschitz equivalent*. A weaker equivalence relation is *quasi-isometry*: two metric spaces (X_1, d_1) and (X_2, d_2) are quasi-isometric if there exist maps $f_1 : X_1 \longrightarrow X_2$ and $f_2 : X_2 \longrightarrow X_1$ such that $\sup_{x\in X_1} d_1(x, f_2 \circ f_1(x)) < \infty$, $\sup_{x\in X_2} d_2(x, f_1 \circ f_2(x)) < \infty$, and there exists $C > 1$ such that $d_2(f_1(x), f_1(y)) \leq Cd_1(x, y)+C$ for all $x, y \in X_1$, and $d_1(f_2(x), f_2(y)) \leq Cd_2(x, y)+C$ for all $x, y \in X_2$ (maps between metric spaces satisfying the latter properties are called *large-scale Lipschitz*).

Quasi-isometry is the natural isomorphism relation between metric spaces in the context of geometric group theory. For example, the fundamental group of a compact Riemannian manifold is quasi-isometric to its universal covering. More generally, if a group G acts on a geodesic metric space X by isometries so that the action is *proper* (for every $x \in X$ and $R > 0$, the set of elements $g \in G$ such that $d(x, g(x)) < R$ is finite) and *cobounded* (there exists a bounded subset of X intersecting every G-orbit), then G is quasi-isometric to X.

A group theoretic notion is considered geometric if it is invariant under quasi-isometries. It is not always easy to decide which group theoretic properties are geometric. For example, the fact that the property to contain a finite index subgroup isomorphic to $\mathbb{Z}^d$ (for a given d) is geometric is rather nontrivial (see [Sha04]). The property of containing a nilpotent subgroup of finite index is a geometric property, by M. Gromov's theorem on groups of polynomial growth [Gro81], but the same statement for solvable subgroups is not true [Dyu00].

Quasi-isometry is the isomorphism relation in a naturally defined *large-scale* category. The objects of the category are metric spaces, the morphisms are large-scale Lipschitz maps $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ up to the equivalence: $f_1 \sim f_2$ if $\sup_{x\in X_1} d_2(f_1(x), f_2(x)) < \infty$.

Let us say that a relation $R \subset X \times X$ on a metric space (X, d) is *bounded* if $\sup_{(g_1,g_2)\in R} d(g_1, g_2) < \infty$. It is an easy exercise to prove that a map $f : G_1 \longrightarrow G_2$ between finitely generated groups is large-scale Lipschitz if and only if it maps bounded relations on G_1 to bounded relations on G_2. The same statement is true for graphs and, more generally, for *quasigeodesic* metric spaces (for example for Riemannian manifolds). It is not true for general metric spaces, so that it is natural to enlarge the large-scale category to the *coarse category*. Namely, a map $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ between metric spaces is *coarsely Lipschitz* if there exists a nondecreasing function $\Phi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $d_2(f(x), f(y)) \leq \Phi(d_1(x, y))$ for all $x, y \in X_1$.

Note that a relation $R \subset G \times G$ is bounded if and only if the set $\{g_1^{-1}g_2 \ : (g_1, g_2) \in R\}$ is finite. This characterization uses neither the word metric nor the fact that G is finitely generated, so that the notion of a bounded relation can be naturally generalized to infinitely generated groups. The properties of the set of all bounded relations can be axiomatized, which leads to the notion of a *coarse structure*; see [Roe03].

The idea that the notions and methods of geometric group theory are interesting and important—not only in the class of finitely generated groups, but also for the much larger class of locally compact groups—became popular only relatively recently. But it has already proved its fruitfulness; see, for example, the papers [CCMT15, CT17].

Many geometric notions (i.e., notions invariant under the isomorphisms in the large-scale or coarse categories) can be naturally extended to the class of locally

compact groups. Instead of using a word metric, one can use the notion of an *adapted pseudometric*. Here a *pseudometric* is a nonnegative symmetric function $d(x, y)$ satisfying the triangle inequality and such that $d(x, x) = 0$. Informally, it is a metric that allows zero distance between different points. A pseudometric d on a locally compact group G is *adapted* if it is left-invariant, its balls have compact closures, and every point has a neighborhood of finite diameter. We do not require that the metric is continuous with respect to the topology of G (as we are interested in large-scale and coarse rather than topological properties). A metric is said to be *geodesically adapted* if it is adapted and quasigeodesic (i.e., quasi-isometric to a geodesic metric space).

One can show [CdlH16, Proposition 1.D.1] that for a σ-compact group G any two adapted metrics are coarsely equivalent. In fact, the corresponding coarse structure is defined by the condition that a relation $R \subset G$ is bounded if and only if the set $\{g^{-1}h \ : \ (g, h) \in R\}$ has compact closure. Moreover, by [CdlH16, Proposition 1.D.1], a group is σ-compact if and only if it has an adapted pseudometric.

This approach nicely extends the techniques of geometric group theory to a much larger domain. For instance, one can consider such notions as coarse connectedness, being coarsely or large-scale geodesic, coarse simple connectedness, etc. A space X is said to be coarsely connected if there exists $R > 0$ such that for any two points $x, y \in X$ there exists a sequence $x = x_0, x_1, x_2, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq R$ for all i. The name "coarsely connected" is justified by the fact that a space is coarsely connected if and only if it is coarsely equivalent to a connected metric space. Many notions of asymptotic group theory also can be generalized to the locally compact setting: growth functions, asymptotic dimension, amenability, etc.

The coarse and large-scale properties of metric spaces have natural interpretation in the class of locally compact groups. For example, a locally compact group is compactly generated if and only if it is coarsely connected. Similarly, a locally compact group is compactly generated if and only if it is coarsely geodesic with respect to any adapted pseudometric. In fact, the word metric (defined with respect to a compact generating set) can be used in the case of compactly generated groups. Note, however, that the word metrics usually do not agree with the topology on the group and are sometimes less natural than other quasigeodesic metrics (for example, in the case of Lie groups). The analogy with classical geometric group theory is complete: any two compact generating sets define quasi-isometric metrics on the group. Moreover, any two pseudometrics with respect to which the group is quasigeodesic are quasi-isometric to each other [CdlH16, Proposition 4.B.10].

On the other hand, there are many new notions and topics in large-scale geometry of locally compact groups, not present in the discrete case. Sections 4.D and 4.E of [CdlH16] give examples of such topics (locally elliptic groups and capped subgroups).

A finitely generated group has a finite presentation by generators and defining relations if and only if it is quasi-isometric to a simply connected metric space. In one direction it follows from the well known fact that a finitely presented group is a fundamental group of a finite simplicial complex. The universal covering of the complex will be quasi-isometric to the group and simply connected.

An analogous fact is true for locally compact groups: a σ-compact, locally compact group G is coarsely simply connected if and only if it is compactly presented. It follows that being compactly presented is a property of the coarse equivalence

class of the group. Here a *compact presentation* is given by a compact generating set S and a set R of defining relations such that the lengths of the elements of R as words in S are bounded. The definition of *coarse simple connectedness* is a natural generalization of simple connectedness to the coarse category. In particular, a coarsely geodesic space is coarsely simply connected if and only if it is coarsely equivalent to a simply connected space. For example, every connected and simply connected Lie group is compactly presented. Many interesting examples of compactly presented groups and their properties are discussed in [CdlH16, Chapter 8]. It is interesting that compact presentability of topological groups can be used to prove finite presentability of discrete groups; see [CdlH16, Section 1.E].

The monograph *Metric geometry of locally compact groups* by Yves Cornulier and Pierre de la Harpe is the first book devoted to large-scale geometry of locally compact groups. It is an excellent introduction into the subject, with detailed exposition of all fundamental theorems and a great variety of examples and applications. It also contains new results, published for the first time in the monograph (for example, Section 4.D on locally elliptic groups, and Section 8.C on a topological version of the Bieri–Strebel splitting theory).

The book consists of eight chapters. Chapter 1 is an introduction with an overview of the main topics of the book. Chapter 2 is a concise but comprehensive introduction to classical results on topological and structural properties of locally compact groups. Chapter 3 describes the coarse and large-scale categories of pseudometric spaces and introduces the main large-scale properties: coarse connectedness, coarse and large-scale geodesicity, coarsely ultrametric spaces, asymptotic dimension, growth, and amenability. Chapter 4 describes locally compact groups as objects of the coarse (in the σ-compact case) and the large-scale (in the compactly generated case) categories. The main results of this chapter are existence of *adapted* and *geodesically adapted* pseudometrics on locally compact groups, and how coarse and large-scale structures on groups are inherited from the space on which the group acts geometrically. Other subjects of Chapter 4 are locally elliptic groups, capped subgroups, amenability and geometric amenability of locally compact groups.

Chapter 5 contains a wide variety of examples of compactly generated locally compact groups. This class is a natural generalization of the class of finitely generated groups, i.e., of the main objects of study of classical geometric group theory. Examples discussed in Chapter 5 include connected groups, abelian and nilpotent groups, Lie groups, algebraic groups over locally compact fields, isometry groups of metric spaces, and many more.

The culmination of the book falls within Chapters 6–8 and is devoted to the generalization of the notion of a finitely presented group: coarsely simply connected spaces and compactly presented groups. As was already mentioned above, compactly presented groups constitute an interesting and natural class of locally compact groups.

The book is written in a clear and accessible style, with many motivating examples and applications.

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