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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ISABELLE GALLAGHER

MR1014927 (90k:82045) 82A40; 35Q20, 45K05, 76P05

DiPerna, R. J.; Lions, P.-L.

On the Cauchy problem for Boltzmann equations: global existence and weak stability.

Annals of Mathematics. Second Series **130** (1989), *no.* 2, 321–366.

This remarkable paper gives the first existence proof of large amplitude global solutions to the Boltzmann equation. So far, only classical L^∞ global solutions having small amplitude either near equilibria [the reviewer, Proc. Japan Acad. **50** (1974), 179–184, MR0363332] or near vacuum [R. Illner and M. Shinbrot, Comm. Math. Phys. **95** (1984), no. 2, 217–226; MR0760333] were known. Here, weak L^1-solutions of arbitrary amplitude are constructed for initial data satisfying only the physically natural condition that the total mass, energy and entropy be finite. Thus, this paper establishes a foundation of the L^1-theory of the Boltzmann equation.

The difficulty in the L^1-theory is the lack of continuity properties of the nonlinear collision operator. What the authors essentially show in this paper is that this can be overcome by introducing a new type of the weak solution, called the renormalized solution. The definition of the renormalized solution involves a new normalization of the Boltzmann equation so that the nonlinear term becomes, roughly speaking, continuous, which ensures weak L^1 stability in the sense that a sequence of classical L^1 solutions satisfying only a priori bounds for the physical quantities mentioned above converges weakly in L^1 to a renormalized solution. This is impossible if one requires the limit to be a classical solution. Applying this stability result to approximate solutions constructed in an appropriate manner, the authors prove the global existence of renormalized solutions.

The ideas for proving the stability are also nice. In particular, the entropy equality, which is a fundamental equality in the celebrated Boltzmann H-theorem but has never been used effectively as an a priori estimate, is used here in a subtle way, combined with recent compactness results for velocity averages, to prove the continuity of the nonlinear term. Also, supersolutions and subsolutions are defined to study the limit, the proof being strongly measure-theoretic.

The uniqueness and regularity of renormalized solutions are important open problems.

Seiji Ukai

From MathSciNet, October 2018

MR1115587 82C40; 82C31

Bardos, Claude; Golse, François; Levermore, David

Fluid dynamic limits of kinetic equations. I. Formal derivations.

Journal of Statistical Physics **63** (1991), *no.* 1-2, 323–344.

The authors discuss the connection between kinetic theory (Boltzmann, Fokker-Planck, BGK-like equations) and macroscopic fluid dynamics (Euler, Navier-Stokes

equations). Formal limits are systematically derived and some rigorous results are given concerning the validity of these limits. The ranges of parameters are described for which these equations provide a good approximation to the solution of the kinetic equation. Three theorems are proved concerning the existence and uniqueness of solutions of Navier-Stokes equations. The authors demonstrate that the connection between kinetic and macroscopic fluid dynamics results from two types of properties of the collision operator $C(F)$: (i) conservation properties and an entropy relation that implies that the equilibria are Maxwellian distributions for the zeroth-order limit; (ii) the derivative of $C(F)$ satisfies a formal Fredholm alternative with a kernel related to the conservation properties of (i). Properties (i) are sufficient to derive the compressible Euler equations; properties (ii) are used to obtain Navier-Stokes equations—they depend on a more detailed knowledge of the collision operator.

Andrzej Fuliński

From MathSciNet, October 2018

MR1213991 82C40; 76A02, 76D05, 76P05

Bardos, Claude; Golse, François; Levermore, C. David

Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation.

Communications on Pure and Applied Mathematics **46** (1993), *no.* 5, 667–753.

This is a further discussion of the connections between kinetic theory (Boltzmann equation) and macroscopic fluid dynamics (Navier-Stokes equations). The authors deal with solutions of the Boltzmann or Navier-Stokes equations in the weakest possible sense that is compatible with basic physical properties, such as conservation laws or the entropy inequality. It is shown that any properly scaled sequence of renormalized solutions of the Boltzmann equation due to R. J. DiPerna and P.-L. Lions [Ann. of Math. (2) 130 (1989), no. 2, 321–366; MR1014927] has fluctuations that converge to an infinitesimal Maxwellian with fluid variables that satisfy the incompressibility and Boussinesq relations. However, such solutions lack local conservation laws of momentum and energy. If an assumption of local momentum conservation is added, the momentum densities globally coverge to a solution of the Stokes equation.

Part I has been reviewed [MR1115587].

Andrzej Fuliński

From MathSciNet, October 2018

MR1340046 76C05; 35-02, 35Q35, 76-02

Chemin, Jean-Yves

Fluides parfaits incompressibles.

Astérisque No. 230 (1995), 177 pp.

The theory of the Euler equation for an incompressible inviscid fluid has been revived in the early nineties, mainly by researchers from the École Polytechnique. After a 28-year-long wake, two main open questions were solved in 1991: the persistence of smooth vortex patches [J.-Y. Chemin, in *Séminaire sur les Équations aux Dérivées Partielles, 1990–1991*, Exp. No. XIII, 11 pp., École Polytech., Palaiseau,

1991; see MR1131573; Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 517–542; MR1235440] and the existence of solutions for vortex sheet initial data [J.-M. Delort, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 1, 85–88; MR1086507; J. Amer. Math. Soc. **4** (1991), no. 3, 553–586; MR1102579]. Since these results deal with planar motions, the last big problem to elucidate is whether smooth solutions of 3D flows survive, or blow up in finite time.

The book under review has two main goals, namely to present the classical theory (Wolbiner, Yudovich and others) on the one hand, and the new theory on the other hand. The true quality of this text is that it unifies the past and the present by an intensive use of Sobolev and Hölder estimates of the paraproduct given by the Littlewood-Paley decomposition (see Chapters 2 and 3). In these, the space C^1_* plays the role of a Hölder space instead of C^1. Although the former is larger than C^1, it still fits well enough with the theory of ODEs so that a flow may be defined for such velocity fields. Thus Lagrangian properties may be addressed.

The Euler equation is introduced in Chapter 1 as a consequence of the least action principle. This is a true mathematical definition, which is not fully convincing from a mechanical point of view for several reasons. First, it cannot be generalized to real (that is, viscous) fluids. Secondly, although it may be adapted to the compressible case, the variational principle does not distinguish a time arrow, so that it cannot select a physically admissible solution from among others. One might argue that incompressible inviscid flows do not need an "entropy condition", since they admit unique global smooth solutions, but this is not entirely true: the 3D flows might not be smooth beyond some finite time; even in 2D, vortex sheet initial data would yield very weak solutions for which we are not able to prove uniqueness.

Nevertheless, regarding the mathematical scope of this book, this definition of a fluid is certainly acceptable. It is followed in Chapter 2 by a very neat presentation of the Littlewood-Paley theory. This part could be of great help, even for graduate students. It ends with a study, via paradifferential calculus, of the quadratic operator $v = \text{velocity} \overset{\Pi}{\mapsto} \text{pressure } = p := -\Delta^{-1}\partial_i\partial_j(v_iv_j)$.

Chapter 3 is devoted to estimates of the velocity in terms of estimates of the vorticity $\Omega(v)$. A special emphasis is given to the notion of tangential regularity with respect to a geometrical structure, anticipating the vortex patch problem.

Chapter 4 presents a new approach to Wolibner's classical solutions (1933). The Euler equation is seen as an abstract evolution equation with a quadratic operator Π satisfying tame estimates. In dimension two, one also assumes the transport equation for the vorticity, but in all cases, one does not make use of energy conservation.

Chapter 5 considers first Yudovich's theory [V. I. Yudovich, Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066; MR0158189] for initial data with bounded vorticity in 2D. The solution is still unique and its structure is preserved. However, the Hölder regularity of the flow may decay as time increases and an explicit example shows (Section 5.3) that it does decay in general. The especially important Section 5.5 proves the persistence of 2D vortex patches. This result, due to the author (see above) and independently to P. Serfati [C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), no. 6, 515–518; MR1270072], seemed unbelievable when announced in 1991. Indeed, numerical experiments of N. J. Zabusky [J. Comput. Phys. **30** (1979), no. 1, 96–106; MR0524163] suggested that singularities of the boundary of the patches would develop, presumably in finite time. A. J. Majda [Comm. Pure Appl. Math.

39 (1986), no. S, suppl., S187–S220; MR0861488] even conjectured, concerning the boundary of piecewise constant vortex patches, that there are smooth initial curves such that the curve becomes nonrectifiable in finite time.

In the same (wrong) direction, S. Alinhac [J. Funct. Anal. **98** (1991), no. 2, 361–379; MR1111574] studied a quadratic approximation of the equation governing the evolution of the boundary and found some evidence of finite time breakdown. Thus it is fair to say that the author's result is one of the biggest achievements of the last decade in the mathematical theory of fluid flows.

The next chapter is devoted to the problem of vortex sheets in two space dimensions. Intermediate results were obtained by R. Di Perna and Majda [Comm. Pure Appl. Math. **40** (1987), no. 3, 301–345; MR0882068]. It is the weakest of the 2D theories, in the sense that it provides only the existence of a weak solution and says nothing about regularity and uniqueness. Since the Birkhoff-Rott equation, which formally governs the evolution of Lipschitz vortex sheets, is linearly ill-posed, we do not expect the geometric structure of the solution to persist for positive time in general. In particular, the flow should not be well-defined as a measure-preserving one-to-one mapping.

The key point in Delort's analysis is the study of a singular integral. It assumes that the singular part of the initial vorticity is nonnegative (or nonpositive). This severe restriction seems to be justified by numerical experiments of R. Krasny, but who knows?

Chapters 7 (wave front, Gevrey regularity) and 9 (vortex patches with non-smooth boundaries) are less fundamental. However, Chapter 8 presents recent results about the time regularity of the flow which are far from obvious. In short, the flow is analytic with respect to time in the classical theory (velocity in C^r, $r > 1$). It is still Gevrey-3 in Yudovich's theory ($\Omega(v)$ bounded). Nevertheless, the space regularity is much weaker, since the velocity field is not smoothed out. The results are due to the author [J. Math. Pures Appl. (9) **71** (1992), no. 5, 407–417; MR1191582].

Overall, this book is a wonderful piece of mathematics and offers an almost complete overview on the field of the mathematical theory of incompressible inviscid fluids (although one might wish to learn about initial-boundary value problems). It clearly shows how big the difference is between the 2D and the 3D cases (the reader should be aware of the fact that a few theorems and sections make sense only in 2D, although the author does not say so). This book fills a wide gap. As far as the reviewer knows, there has not existed a detailed overview on this topic. Such a new book in a classical field is a rare event and most of the researchers in theoretical fluid mechanics will want to have it in their private libraries.

Denis Serre

From MathSciNet, October 2018

MR1755865 (2002b:76028) 76D03; 35A05, 35Q30, 76D05

Amann, Herbert

On the strong solvability of the Navier-Stokes equations.

Journal of Mathematical Fluid Mechanics **2** (2000), *no.* 1, 16–98.

As is known, the problem of unique solvability "in the large" for general three-dimensional nonstationary Navier-Stokes equations remains unsolved, and constitutes one of the most challenging questions of the last millennium. Some essential

progress has been achieved recently in this field, mainly with the semigroup approach. However, each author has his own solution, and the spaces where uniqueness is proven are not comparable to each other; in fact "there are infinitely many solutions due to the possible choices of underlying spaces" [see O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow* (Russian), Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Gordon and Breach, New York, 1969; MR0254401].

The article under review starts with a critique of most of those papers where existence of a local (or global) solution is proven in a "suitable space" (in general different in each paper) while uniqueness is proven under additional artificial restrictions on that solution. One of the aims of the paper is to fill this gap. More generally, the present paper is an attempt to show that there is an analog to most of the above theorems; this goal is reached by the use of refined topological and functional tools.

Let us begin with some definitions of functional spaces used throughout the paper: (1) $L_{s,\sigma}$ denotes the projection of the Lebesgue space L_s into the subspace of solenoidal vector fields orthogonal to gradient fields; (2) $L_r((0,T);L_{s,\sigma})$ is the Bochner space of functions in $\Omega\times(0,T)$ having the spatial norms $L_{s,\sigma}$ summable in time with exponent r; (3) $\mathcal{D}([0,T);D_\sigma)$ is the space of infinitely differentiable solenoidal vector functions in $\Omega\times[0,t)$; (4) S is the Stokes operator $-\nu\Delta$; (5) n^s_q are the little Nikol′skiĭ spaces, that is, the closures of H^s_q in the Besov spaces $B^s_{q,\infty}$, and $n^s_{s,0,\sigma}$ are their projections on solenoidal fields. Another tool is the concept of finest locally convex topology on the union $\bigcup E_\alpha$ of a family of Banach spaces $\{E_\alpha\colon \alpha>\alpha_0;\ E_{\alpha_0}\to E_\alpha\}$ such that each of the natural inclusions $E_\beta\to\bigcup E_\alpha$ is continuous. Thus, the direct limit of the family $\{E_\alpha\colon\ \alpha>\alpha_0\}$ is defined as $\bigcup E_\alpha=\lim_\alpha E_\alpha$. This topology is applied in particular to the little Nikol′skiĭ spaces, denoted $n^{-1+m/r}_{r,0,\sigma}$, where $m/3<r<\infty$. Finally, we recall the concept of spaces well suited for the Navier-Stokes equations, used in [M. Cannone and Y. F. Meyer, Methods Appl. Anal. **2** (1995), no. 3, 307–319; MR1362019].

Let us recall various definitions of solutions to the Navier-Stokes system: The very weak solution in the space $L_r(0,T;L_{s,\sigma})$, used by E. B. Fabes, B. F. Jones and N. M. Rivière [Arch. Rational Mech. Anal. **45** (1972), 222–240; MR0316915]. The mild solution in E, where E is a Banach space of distributions on which the Stokes semigroup $\{\exp^{-tS}\colon\ t\ge0\}$ is strongly continuous and the integral

$$\int_0^t \exp^{-(t-\tau)S}[-P(u\cdot\nabla u)],\quad u\in E,$$

is well defined. The maximal strong solution v, that solution for which there exists a maximal existence time t^+ such that $v\in C([0,t^+),n^{-1}_{\infty,0,\sigma})$, used in this paper. The strong q-solution on $[0,t)$, also used in the present paper.

Actually, the author proves the existence of a unique maximal strong solution, for rough initial data, and the coincidence of various concepts of generalized solutions; moreover he obtains some improvements over previous uniqueness and smoothness theorems. For simplicity, the domains of motion, called "standard domains", are either $\mathbf{R}^m$, $m\ge2$, or domains with compact boundaries, or half-spaces.

The main results of the paper are the following.

Theorem 1. Suppose $m < q \le r < \infty$ and $v^0 \in n^{-1+m/q}_{q,0,\sigma}$. There exists a unique maximal solution of the Navier-Stokes equations in $C((0,t^+), H^2_{r,0,\sigma}) \cap C^1((0,t^+), L_{r,\sigma})$ satisfying

$$\lim_{t\to 0} v(t) = v_0 \quad \text{in } n^{-1+m/q}_{q,0,\sigma},$$

$$\lim_{t\to 0} t^{(1-m/q)/2} v(t) = 0 \quad \text{in } L_q.$$

Theorem 2. Suppose $m < q \le r < \infty$ and $v^0 \in H^{-1+m/q}_{q,0,\sigma}$. (i) There exists a unique maximal strong solution of the Navier-Stokes equations satisfying

$$\lim_{t\to 0} v(t) = v^0 \quad \text{in } H^{-1+m/q}_q$$

and, if $q > m$,

$$\lim_{t\to 0} t^{(1-m/q)/2} v(t) = 0 \quad \text{in } L_q.$$

It is smooth for $0 < t < t^+$. (ii) Let $\mathcal{F}^s_{q,0,\sigma}$ be one of the spaces $H^s_{q,0,\sigma}$, $B^s_{q,r,0,\sigma}$, $n^s_{q,0,\sigma}$, $1 \le r < \infty$, for some $s \in ((-1+m/q)_+, 2]$; then

$$\lim_{t\to 0} v(t) = v^0 \quad \text{in } \mathcal{F}^s_{q,0,\sigma},$$

provided $v^0 \in \mathcal{F}^s_{q,0,\sigma}$. (iii) If $q \ge m$ then $v \in L_r((0,t), L_s)$, with $t < t^+_q$, $r \in [2,\infty]$ and $s \in [m,\infty)$, and $2/r + m/s = m/q$. (iv) Given $t > 0$, there exists $R > 0$ such that $t^+(v^0) > t$ for $\|v^0\|_{n^{-1+m/q}_{q,0,\sigma}} \le R$.

In two theorems, the author proves the equivalence of various definitions of solution; let us call them $v(\cdot, v^0)$. Concerning the blow-up in time, the following holds. Theorem 3. Suppose $m/3 < q < \infty$ and $v^0 \in H^{-1+m/q}_{q,0,\sigma}$. (i) If $t^+ < \infty$ then

$$\lim_{t\to 0} \|v(t)\|_{H^s_{r,0,\sigma}} = \infty$$

for all $r > m$ with $r \ge q$ and every $s > -1 + m/r$. (ii) Suppose $r > m$ with $r \ge q$ and $-1 + m/r < s < 0$. Then

$$\|v(t)\|_{H^s_{r,0,\sigma}} \ge c(t^+ - t)^{-(s+1-m/r)/2},$$

where $0 < t^+ - t \le 1$, and c is independent of v^0.

These theorems are proved for standard domains.

We now turn our attention back to the basic problem of uniqueness of the global weak Leray-Hopf solution. The following theorem guarantees uniqueness and smoothness on the maximal existence interval of strong solutions $v(\cdot, v^0)$; the novelty relies on the following uniqueness result.

Theorem 4. Suppose Ω is a standard domain and $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$, $q \ge m$. Then $v := v(\cdot, v^0)$ is a weak solution on $[0,T]$, $T \in (0,t^+)$. It belongs to $C([0,t^+), L_2)$ and satisfies the strong energy identity

$$\|v(t)\|^2_{L_2} + 2\nu \int_s^t \|\nabla v(\tau)\|^2_{L_2} d\tau = \|v(s)\|^2_{L_2}, \quad 0 \le s < t < t^+.$$

If u is any Leray-Hopf weak solution then $v(\cdot, v^0) \subset u$. In particular, u is smooth on $(0,t^+)$.

We would like to end with some comments on the results achieved, and on the bibliography in this paper.

The results are achieved within the theory of semigroups, based essentially on fine estimates on the Stokes operator. This approach utilizes interpolation-extrapolation techniques to yield sharp results on the convective term.

The present paper is compared with those papers that use the same tools, and from this perspective the results here improve or generalize the previous ones. In particular, the author extends the results of Cannone and Meyer [op. cit.] to domains with boundary.

The article under review provides a detailed description of the spaces of initial data for the velocity; these spaces are very general. In particular, both $H^{-1+m/q}_{q,0,\sigma}$ and $n^{-1+m/q}_{q,0,\sigma}$ can have negative exponent for q sufficiently large. The author provides a generalization of the spaces where blowup can occur; in particular they can be Lebesgue spaces. The equivalence of various definitions of weak solutions appears to be very useful. A crucial point improved in this paper consists in the proof of natural extensions of Sobolev type embedding theorems for the spaces $H^{-1+m/q}_{q,0,\sigma}$ or $n^{-1+m/q}_{q,0,\sigma}$ in the presence of a boundary.

The author proves the existence of a unique extension of the Helmholtz projector to negative spaces; this allows him to obtain sharp results for the nonlinear convection term which improve known continuity estimates. The new uniqueness theorem for very weak q-solutions is proved in a rather simple way.

The article under review quotes almost a hundred articles in its bibliography, and fills a large lacuna in the existence of solutions of the Navier-Stokes equations.

The article provides a valuable insight into a very modern field, sheds new light on the resolution of a key problem, and is very clearly written. In particular, the introduction contains the outline of the proof and is perfectly accessible even to the non-expert in this area.

Mariarosaria Padula

From MathSciNet, October 2018

MR1842343 (2002m:76085) 76P05; 35F20, 35Q35, 76A02, 76D05, 82C40

Lions, P.-L.; Masmoudi, N.

From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II.

Archive for Rational Mechanics and Analysis **158** (2001), *no.* 3, 173–193, 195–211.

These two papers are the first ones of a series devoted to the mathematical derivation of incompressible fluid mechanics from the Boltzmann equation in the appropriate scaling when the Mach number and the Knudsen number go to zero. It may be viewed as an extension of previous work of C. Bardos, F. Golse and C. D. Levermore [C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 11, 727–732; MR1054287; J. Statist. Phys. **63** (1991), no. 1-2, 323–344; MR1115587; Comm. Pure Appl. Math. **46** (1993), no. 5, 667–753; MR1213991; C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 3, 323–328; MR1650310]; for a different approach to the same problem, see A. De Masi, R. Esposito and J. L. Lebowitz [Comm. Pure Appl. Math. **42** (1989), no. 8, 1189–1214; MR1029125]. In the first paper the authors show that the global (renormalized) solutions of the Boltzmann equation [R. J. DiPerna and P.-L. Lions, Ann. of Math. (2) **130** (1989), no. 2, 321–366; MR1014927] converge, as the mean free path goes to zero with appropriate scalings, to global (weak) solutions of the incompressible Navier-Stokes equations. The main difference with

respect to the results of Bardos et al. is that the proof in the present paper allows rather general initial conditions and is global in time. However, the authors need to impose, as was done by Bardos et al., various conditions on the solutions of the Boltzmann equation which are not known to hold. In the second paper the latter conditions are somewhat relaxed or even completely suppressed (in the special case of Stokes' linear equations). Similar results are obtained for incompressible Euler equations, under one extra assumption on the behavior of the solutions for large speeds.

Carlo Cercignani

From MathSciNet, October 2018

MR2025302 (2005f:76003) 76A02; 35F20, 35Q30, 76D03, 76D05, 76P05, 82C40

Golse, François; Saint-Raymond, Laure

The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels.

Inventiones Mathematicae **155** (2004), *no.* 1, 81–161.

In 1934, J. Leray [Acta Math. **63** (1934), 193–248; JFM 60.0726.05] established the existence of weak solutions to the incompressible Navier-Stokes equation in the whole space, for any initial velocity field with finite kinetic energy. With this work he hoped to develop a framework in which qualitative properties of solutions for fluid mechanics equations could be studied with a lot of generality (no assumptions of smallness, only physically relevant assumptions such as the finiteness of the total kinetic energy). One of the important contributions in this field was the construction by R. J. DiPerna and P.-L. Lions [Ann. of Math. (2) **130** (1989), no. 2, 321–366; MR1014927] of weak ("renormalized") solutions to the Boltzmann equation. The main result of the paper under review can be summarized informally as follows: Limits of suitably rescaled sequences of DiPerna-Lions solutions are Leray solutions.

This striking result can be considered as the culminating point of fifteen years of efforts by various authors (Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond). It is the first asymptotic theorem relating the Boltzmann and Navier-Stokes equations without any unphysical restriction of size or smoothness, and may be seen as a major event, be it considered from the point of view of the theories of weak solutions of partial differential equations (since it bridges two of the most famous such theories, the ones by Leray and by DiPerna-Lions) or from the point of view of mathematical limits from Boltzmann to hydrodynamics equations (a part of Hilbert's Sixth Problem about the derivation of macroscopic fluid mechanics equations from mesoscopic, or ideally microscopic, models).

The starting point in the paper is the classical Boltzmann equation, describing a dilute gas in which particles interact via binary collisions; the reader can consult the long survey by the reviewer [in *Handbook of mathematical fluid dynamics, Vol. I*, 71–305, North-Holland, Amsterdam, 2002; MR1942465] for a mathematically oriented introduction to this model and many references. In the version used by the authors, the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f,f), \quad t \geq 0, \quad x \in \mathbb{R}^3, \quad v \in \mathbb{R}^3,$$

where t stands for time, x for position, v for velocity, and the unknown $f = f(t, x, v)$ is a nonnegative function having the meaning of a time-dependent particle density in phase space. Here the parameter $\mathrm{Kn} > 0$ is the Knudsen number, which can be defined heuristically as the inverse of the mean number of collisions that one particle undergoes in a unit of time. Finally, Boltzmann's collision operator $Q(f, f)$ is defined by

$$Q(f,f)(v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, B(v - v_*, \sigma)\big[f(v')f(v'_*) - f(v)f(v_*)\big],$$
$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \quad (\sigma \in S^2).$$

Here the nonnegative function $B(v - v_*, \sigma)$ is Boltzmann's collision kernel, depending on the microscopic interaction. For technical reasons it is assumed by the authors that B is bounded from above and below by positive constants; it is also assumed that the solutions of certain linear equations involving the linearized Boltzmann operator have at most polynomial growth. This latter assumption is known to be satisfied at least when B only depends on the angle between $v - v_*$ and σ (Maxwellian collision kernel), and it is reasonable to conjecture that it holds true with much more generality.

Here are a few sketchy explanations about the physical context of this study. When the Knudsen number is very small, collisions are expected to occur very frequently and drive the solution very close to a time-dependent local Maxwellian state,

$$M(x,v) = \rho(x)\frac{e^{-|v-u(x)|^2/(2T(x))}}{(2\pi T(x))^{3/2}},$$

where ρ, u and T respectively have the meaning of a density, a mean velocity field and a temperature. Thus, in this regime it should be possible to approximate solutions of the Boltzmann equation by a hydrodynamic equation. The rigorous justification of this creed is an extremely difficult problem, mostly open. The Boltzmann equation is a model for a compressible gas, and its natural hydrodynamical limit is the compressible Euler equation. However, incompressible models can also occur as approximations of the Boltzmann equation in a regime where not only is the Knudsen number very small, but also the solution is a very small perturbation of the equilibrium, which is the global Maxwellian

$$M(x,v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}}.$$

This is the situation studied by the authors: they introduce a sequence $\varepsilon_n \to 0$ playing the role of a vanishing Knudsen number, and introduce a sequence $(f_n)_{n\in\mathbb{N}}$ such that each f_n is a solution to the Boltzmann equation (in the sense of DiPerna and Lions) with Knudsen number ε_n; then they set

$$f_n(t,x,v) = M(x,v)[1 + \varepsilon_n g_n(\varepsilon_n t, x, v)].$$

The problem is to study the asymptotic behavior of g_n, which can be seen as a (rescaled) fluctuation of f_n. The time-rescaling by a factor ε_n is in some sense compulsory, since the hydrodynamical evolution in short time is dominated by linear acoustics, and the Navier-Stokes equation only arises as a longer-time correction.

Before stating the result more precisely, we feel that further explanations and notation will be useful. To ensure that g_n is of size $O(1)$ (in the distribution sense),

it is assumed that the relative H functional (or Kullback information) of f_n with respect to the equilibrium M is of the order ε_n^2:

$$\int \left(\frac{f_n}{M} \log \frac{f_n}{M} - \frac{f_n}{M} + 1 \right) M \, dv \, dx = O(\varepsilon_n^2).$$

By a compactness argument, combined with a variant of Boltzmann's H Theorem, the sequence Mg_n converges weakly, up to extraction of a subsequence, to Mg, where g is a fluctuation of the Maxwellian,

$$g(t,x,v) = \rho(t,x) + u(t,x) \cdot v + \theta(t,x) \left(\frac{|v|^2 - 3}{2} \right).$$

By integrating g_n against the test functions 1, v and $|v|^2/2$ with respect to the reference measure $M(x,v)\,dv$, one defines functions $\rho_n(t,x)$ (scalar), $u_n(t,x)$ (vector-valued) and $\theta_n(t,x)$ (scalar), which can be seen as fluctuations in the density, velocity and temperature fields, respectively. Again, up to extraction of a subsequence, one may assume that they converge weakly to $\rho(t,x)$, $u(t,x)$ and $\theta(t,x)$. The main result in this paper ensures that these limit fields obey the incompressible Navier-Stokes equation. More precisely,

- the limit velocity field $u(t,x)$ is a weak (Leray) solution to the incompressible Navier-Stokes equation,

$$\frac{\partial u}{\partial t} + u \cdot \nabla_x u + \nabla p = \nu \Delta_x u,$$

  where p is the pressure, $\nabla_x \cdot u = 0$ (incompressibility relation) and $\nu > 0$ is a viscosity, which can be computed in terms of the collision kernel;
- the limit temperature $\theta(t,x)$ is advected by the flow and dissipated by the heat conductivity κ (Fourier law),

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla_x \theta = \kappa \Delta_x \theta,$$

  where, again, κ can be computed in terms of the collision kernel;
- the limit density is determined from the limit temperature via the Boussinesq relation, $\rho + \theta = 0$.

It is part of the results of the authors that the incompressibility condition and the Boussinesq identity will be satisfied for positive time even if they do not hold true initially. In that case, the initial data for the limit equation should not be the limits of (ρ_n, u_n, θ_n) at time 0, but rather the limit of

$$\left(\frac{2}{5}\rho_n - \frac{3}{5}\theta_n, P u_n, \frac{3}{5}\theta_n - \frac{2}{5}\rho_n \right)$$

at time 0, where P is the Leray projector on divergence-free vector fields.

Good note should be taken that even if the authors only consider perturbations of equilibrium, their result is definitely "in the large": in the end, solutions of the Navier-Stokes equation with arbitrarily large velocity field (only assumed to have finite kinetic energy) are obtained.

More precise statements can be found in the paper itself; one can also find there a thorough discussion of the history of the results and a very clear, synthetic presentation of the new tools which led to the solution. Here below are some details worth noting.

The study of the behavior of DiPerna-Lions solutions of the Boltzmann equation in the small Knudsen number limit was started by Bardos, Golse and Levermore in the early nineties [C. Bardos, F. Golse and C. D. Levermore, Comm. Pure Appl. Math. **46** (1993), no. 5, 667–753; MR1213991]. They set up the problem in a clear way, studied the various scalings, and showed how to obtain weak compactness by entropy estimates and a crucial strong compactness in the x variable by the technology of velocity-averaging lemmas. More generally, they pushed the theory as far as they could, and stumbled on three well-identified main problems:

(i) All the proofs of hydrodynamical limits, formal or rigorous, relied crucially on the local conservation laws (mass, momentum, kinetic energy), but these conservation laws are still an open problem for the Boltzmann equation.

(ii) The existence of fast acoustic waves prevented the strong compactness, because of rapid time-oscillations (sound propagation), and passing to the limit using only weak compactness looked challenging.

(iii) Some key equi-integrability estimates were lacking, in particular the equi-integrability of the family $|v|^2 g_n^2/(2+\varepsilon_n g_n)$.

Bardos, Golse and Levermore showed that, roughly speaking, the resolution of these three problems would make it possible to pass to the limit. Additional problems were the lack of high-order moment estimates, seemingly turning the control of the heat flux into a hopeless task, so that the discussion was restricted to the momentum equation; and some technical estimates about eigenfunctions, which were known only for certain very particular collision kernels.

All these difficulties have been solved in the last years, starting with the acoustic waves problem. It was understood independently by E. Grenier [C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 6, 711–714; MR1354711] and S. H. Schochet [J. Differential Equations **114** (1994), no. 2, 476–512; MR1303036] that it was possible to "filter" acoustic waves in the incompressible limit of the compressible Navier-Stokes equations, thanks to an adequate "change of coordinates". Later, Lions and N. Masmoudi [C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 5, 387–392; MR1710123] devised a convenient approach to this problem, based on a clever compensated compactness argument, and were able to adapt it to the problem of the Boltzmann equation [Arch. Ration. Mech. Anal. **158** (2001), no. 3, 173–193, 195–211; MR1842343].

The next important progress was the surprising discovery by Golse and Levermore [Comm. Pure Appl. Math. **55** (2002), no. 3, 336–393; MR1866367] that it is not absolutely necessary to derive local conservation laws: under some a priori estimates, local conservation laws may be established *asymptotically* in the limit of small Knudsen number. This was achieved with the help of very clever but very elementary bounds coming ultimately from entropy production estimates and convexity inequalities.

In the paper under review, Golse and Saint-Raymond find a way to bypass difficulty (iii) (the equi-integrability estimate stated above remaining an open problem), and this allows them to present a complete proof of the limit to the incompressible Navier-Stokes system. This achievement was made possible by an impressive refinement of tools that were already known, but have been put here to a new degree of precision and efficiency: in particular,

- several careful decompositions of the fluctuation g_n into a part which is $O(1/\varepsilon_n)$, and a remainder;

- a new, limit case of the velocity-averaging lemma: roughly speaking, if a family of densities is bounded in L^1, is locally uniformly integrable in the velocity variable and satisfies a transport equation with right-hand side bounded in L^1, then it is locally uniformly integrable in all variables.

This is also combined with a clever use of estimates on the bilinear Boltzmann operator, due to Caflisch and Grad. Surprisingly enough, in their proof the authors introduce both an artificial Boltzmann operator (to help control the distance of the distribution to the associated local Maxwellian distribution) and an artificial time variable (for instance to prove the new velocity-averaging lemma via dispersion estimates in the artificial variable). The first idea can be traced back to Lions [J. Math. Kyoto Univ. **34** (1994), no. 2, 391–427, 429–461; MR1284432 (p. 423)], while the application of the second one is reminiscent to the definition of real interpolation trace spaces; yet both tricks have a certain flavor of mystery, and seem to leave room for simplification.

All in all, the proof is of an incredibly high technical level, and the detail of the computations is very difficult to master, although enormous efforts have been made in the style and presentation to facilitate the task of the reader and make the paper as self-contained as possible. These results were presented and commented on, together with other contributions about incompressible limits from the Boltzmann equation, by the reviewer in a Bourbaki seminar [Astérisque No. 282 (2002), Exp. No. 893, ix, 365–405; MR1975186]; in that reference the reader will find much more about the history of the program, bibliographical references and some sketches of proofs which may help in understanding the techniques.

By no means can the results presented here be considered as a final answer to the problem of the hydrodynamic approximation of the Boltzmann equation. On a technical level, the assumption of bounded collision kernel is not physical, and it would be desirable to relax it so as to allow at least the hard spheres kernel $B(v - v_*, \sigma) = |v - v_*|$. This is probably just a (horribly tedious) technical problem. On a more fundamental level, some parts of the proof are definitely nonconstructive; and even if these parts were replaced by constructive arguments, the proof would involve Reynolds numbers that are too large to be realistic, by many many orders of magnitude. Finally, the proof as it stands covers only the case of the whole space. These remarks show that there is still a lot of room for improvement, but do not aim at diminishing the merit of this paper, probably one of the major contributions in mathematical fluid mechanics over the last decade.

Cédric Villani

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MR2407976 (2010h:82086) 82C40; 82-02, 82-06

Rezakhanlou, Fraydoun; Villani, Cédric

Entropy methods for the Boltzmann equation. (English)

Lectures from a Special Semester on Hydrodynamic Limits held at the Université de Paris VI, Paris, 2001.

Lecture Notes in Mathematics, 1916.

Springer, *Berlin*, 2008, xii+107 pp., $44.95, ISBN 978-3-540-73704-9

Contents:

This nice book is based on two courses given, respectively, by Fraydoun Rezakhanlou and Cédric Villani at the Centre Émile Borel of the Institut Henri Poincaré in a special semester organized in the fall term of 2001 by François Golse and Stefano Olla.

The first course, by Villani, deals with the issue of the relaxation to equilibrium of the Boltzmann equation. The second course, by Rezakhanlou, deals with the issue of the Boltzmann-Grad limit for deriving the Boltzmann equation from a many-particle system. The connecting thread through these lectures is the use of entropy. As recalled in the interesting introduction by Golse and Olla, variational characterizations of Maxwellian equilibria of the Boltzmann equation and variational characterizations of chaotic data in many-particle systems both illustrate how entropy can be used as a way to measure the distance to some particular "limit" distribution (an asymptotic limit in the first case, a many-particle limit in the second case).

Let us give more details about the courses. The first course (Villani) is concerned with obtaining a constructive rate of relaxation to equilibrium for the Boltzmann equation by relating the relative entropy and the entropy production functional. The author begins with some motivations for his program of research, going back to Boltzmann, Kac and McKean. He then presents recent results, together with useful reminders from information theory, Cauchy and regularity theory for the Boltzmann equation and Kac's problem for particle systems. Highlights include, first, a very clear presentation of the author's work [C. Villani, Comm. Math. Phys. **234** (2003), no. 3, 455–490; MR1964379] proving Cercignani's conjecture for the spatially homogeneous Boltzmann equation with super-quadratic collision kernels. This conjecture states that the entropy production functional $D(f) = -dH(f)/dt \geq 0$ (with $H(f) = \int f \log f$) and the (increasing) relative entropy functional $-H(f|M) = -(H(f) - H(M))$ with respect to the equilibrium Maxwellian distribution M satisfy

$$D(f) \geq K\, H(f|M) \quad \text{for some positive constant } K.$$

The proof involves a fascinating and clever use of a derivation along the Ornstein-Uhlenbeck semigroup which makes the Landau entropy production functional appear! The second highlight is a short and understandable introduction to the author's work [L. Desvillettes and C. Villani, Invent. Math. **159** (2005), no. 2, 245–316; MR2116276] proving the "almost exponential" relaxation to equilibrium of the full (spatially dependent) Boltzmann equation in a bounded domain, under some a priori moment and regularity assumptions on the solution. Physical and analytical ideas are explained with special emphasis on the key aspect of "how to depart from local equilibria". In the words of Villani, "local equilibria are your worst enemies" when looking for relaxation to global equilibrium. Indeed, entropy and entropy production inequalities proved before in the spatially homogeneous case still play a key role; however, they are degenerate on the set of local equilibria. The key idea introduced by Desvillettes and Villani is to estimate the second time derivative of

some relative entropies, and replace one first-order differential inequality by a set of second-order differential inequalities.

The lecture also discusses many extensions, open questions and perspectives which should be valuable for many researchers in the field of the Boltzmann equation.

The second course (Rezakhanlou) is concerned with the Boltzmann-Grad limit. While the core of the text is somewhat harder to read for non-specialist readers than the previous course, since it goes quickly into detailed intricate computations, it begins with a very clear and illuminating introduction presenting a hierarchy of conjectures and models for studying the Boltzmann-Grad limit. Indeed the author formulates the issue of chaos propagation in probabilistic terms as a law of large numbers, which naturally leads to refined conjectures on small (CLT) and large deviation estimates around the deterministic Boltzmann-Grad limit. He then presents a hierarchy of stochastic models in the spirit of Kac's master equation, with either discrete or continuous space and velocity variables, approximating the Boltzmann equation or the Boltzmann-Enskog equation (with delocalized collision process). He also introduces a ladder of scalings generalizing the usual Boltzmann-Grad scaling.

Rezakhanlou then reviews important recent results he has obtained for the models and conjectures presented earlier and he sketches some proofs of the following highlighted results:

(1) [F. Rezakhanlou, Duke Math. J. **93** (1998), no. 2, 257–288; MR1626003] A central limit theorem for a stochastic discrete velocity model (for scalings one order better than the Boltzmann-Grad scaling and for bounded collision kernels): the proof goes through an estimate "à la Tartar" of the modulus of continuity for the number of collisions for the limit PDE, and a new microscopic analogue of it.

(2) [F. Rezakhanlou, Probab. Theory Related Fields **104** (1996), no. 1, 97–146; MR1367669] The kinetic limit in space dimension 1 for a stochastic discrete velocity model: the proof goes through the introduction of a new microscopic version of the celebrated Bony functional.

(3) [F. Rezakhanlou, Comm. Math. Phys. **248** (2004), no. 3, 553–637; MR2076921] The kinetic limit of stochastic hard spheres towards renormalized DiPerna-Lions solutions (for scalings $(d+1)$-orders better than the Boltzmann-Grad scaling): the key ingredient of the proof is an interesting new result of velocity averaging for the density of (convoluted) empirical measures of N particles following a stochastic collision process, which allows passing to the chaotic limit in the collision part of the equation for these empirical measures.

This book, at the crossroads of probability and partial differential equations, will be useful to all mathematicians interested in entropy methods.

Clément Mouhot

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Saint-Raymond, Laure

Hydrodynamic limits of the Boltzmann equation. (English)

Lecture Notes in Mathematics, 1971.

Springer-Verlag, *Berlin*, 2009, xii+188 pp., $59.95, ISBN 978-3-540-92846-1

From a physical point of view, we expect that a gas can be described by a fluid equation when the mean free path (Knudsen number) goes to zero. In his sixth problem, on the occasion of the International Congress of Mathematicians held in Paris in 1900, Hilbert asked for a full mathematical justification of these derivations. During the last two decades this problem has attracted a lot of interest.

Let us first give some background about this problem (see Chapters 1 and 2 in the book). The molecules of a gas can be modeled by spheres that move according to the laws of classical mechanics. However, due to the enormous number of molecules to be considered, it is hopeless to describe the state of the gas by giving the position and velocity of each individual particle. Hence, we must use some statistics and instead of giving the position and velocity of each particle, we specify the density of particles $F(x, v)$ at each point x and velocity v. Under some assumptions (rarefied gas, etc.), it was proved by Boltzmann (and Lanford for a rigorous proof in the hard sphere case) that this density is governed by the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = B(F, F). \tag{B}$$

To derive fluid equations from the Boltzmann equation, one has to introduce several dimensionless parameters: the Knudsen number Kn (which is related to the mean free path), the Mach number Ma and the Strouhal number St (which is a time rescaling). With these parameters, one can rewrite the Boltzmann equation as

$$\mathrm{St} \cdot \partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} B(F, F)$$

with $F = M(1 + \mathrm{Ma} \cdot f)$ where M is a fixed Maxwellian. It is worth noting that the Reynolds number Re is completely determined by the relation $\mathrm{Ma} = \mathrm{Kn} \cdot \mathrm{Re}$. Several fluid equations can be derived that depend on these dimensionless parameters: Compressible Euler system, acoustic waves, Incompressible Navier-Stokes-Fourier system, Stokes-Fourier system, Incompressible Euler system, etc. There are several approaches to deal with this problem: the weak compactness method initiated by C. Bardos, F. Golse and C. D. Levermore, asymptotic expansions [see A. De Masi, R. Esposito and J. L. Lebowitz, Comm. Pure Appl. Math. **42** (1989), no. 8, 1189–1214; MR1029125], the energy method [Y. Guo, Comm. Pure Appl. Math. **59** (2006), no. 5, 626–687; MR2172804; erratum, Comm. Pure Appl. Math. **60** (2007), no. 2, 291–293; MR2275331], etc.

This book gives an overview of some of these results and mainly the derivation of the Incompressible Navier-Stokes [F. Golse and L. Saint-Raymond, Invent. Math. **155** (2004), no. 1, 81–161; MR2025302] and Incompressible Euler [L. Saint-Raymond, Arch. Ration. Mech. Anal. **166** (2003), no. 1, 47–80; MR1952079] systems.

After the construction of the renormalized solutions to the Boltzmann equation by R. J. DiPerna and P.-L. Lions [Ann. of Math. (2) **130** (1989), no. 2, 321–366; MR1014927], there was a program initiated by Bardos, Golse and Levermore [J. Statist. Phys. **63** (1991), no. 1-2, 323–344; MR1115587; Comm. Pure Appl. Math. **46** (1993), no. 5, 667–753; MR1213991] to derive incompressible models from the Boltzmann equation. In particular the main objective was to recover the Leray [J. Leray, Acta Math. **63** (1934), no. 1, 193–248; MR1555394; JFM 60.0726.05] global weak solutions of the incompressible Navier-Stokes system starting from the DiPerna-Lions solutions.

There were five main assumptions in their first work:

(1) Because of a problem coming from the rapid time-oscillations of acoustic waves, only the time independent case was considered.

(2) Local conservation laws were assumed even though these are not known to hold for the renormalized solutions.

(3) The lack of high-order moment estimates required the restriction of the discussion to the momentum equation and no heat equation was derived.

(4) A key equi-integrability estimate was assumed on the solutions of the Boltzmann equation. This is due to the fact that the natural space for the Boltzmann equation is $L \log L$ whereas for the Navier-Stokes system the natural space is L^2.

(5) Due to a technical estimate for the inverse of the linearized Boltzmann kernel, only very particular collision kernels were considered.

These five assumptions have been removed one by one in the past two decades:

(1) In [P.-L. Lions and N. Masmoudi, Arch. Ration. Mech. Anal. **158** (2001), no. 3, 173–193, 195–211; MR1842343] the time-oscillating acoustic waves were treated using a compensated compactness type argument coming from the compressible-incompressible limit [P.-L. Lions and N. Masmoudi, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 5, 387–392; MR1710123].

(2)–(3) In [P.-L. Lions and N. Masmoudi, op. cit., MR1842343 (pp. 195–211)], the assumption on the local conservation in the momentum equation was removed, and in [Comm. Pure Appl. Math. **55** (2002), no. 3, 336–393; MR1866367], Golse and Levermore were able to derive the Stokes-Fourier system. The main idea is to recover the moment conservation laws at the limit.

(4) The main breakthrough of [F. Golse and L. Saint-Raymond, op. cit.; MR2025302] was a new L^1 averaging lemma that allows one to prove the key equi-integrability estimate.

(5) In [F. Golse and L. Saint-Raymond, J. Math. Pures Appl. (9) **91** (2009), no. 5, 508–552; MR2517786] the result was extended to hard cutoff potentials satisfying Grad's cutoff assumption and in [C. D. Levermore and N. Masmoudi, Arch. Ration. Mech. Anal. **196** (2010), no. 3, 753–809; MR2644440] it was also extended to both hard and soft potentials. Another important extension was done by D. Arsénio ["From Boltzmann's equation to the incompressible Navier-Stokes-Fourier system with long-range interactions", Arch. Ration. Mech. Anal., to appear], who treated the non-cutoff case.

We also note that the case where the problem is considered in a bounded domain was treated in [N. Masmoudi and L. Saint-Raymond, Comm. Pure Appl. Math. **56** (2003), no. 9, 1263–1293; MR1980855] where Navier and Dirichlet boundary conditions were derived starting from the Maxwell boundary condition.

Chapter 3 of this book presents the main mathematical tools used in dealing with the hydrodynamic limit. In particular several estimates coming from the entropy, the entropy dissipation and Darrozès-Guiraud information are presented. Also the new L^1 averaging lemma is proved.

Chapter 4 deals with the incompressible Navier-Stokes limit using the weak compactness method. In particular the author shows how to combine the ideas from [N. Masmoudi and L. Saint-Raymond, op. cit.; MR1980855] to treat the case of a bounded domain with Maxwell boundary conditions.

Chapter 5 deals with the incompressible Euler limit using the relative entropy method [L. Saint-Raymond, op. cit.; MR1952079].

Finally, Chapter 6 gives a survey of the known results about the compressible Euler limit. It is worth noting that if we are interested in starting from the renormalized solutions then none of the methods used in the incompressible case can be adapted. The author gives some open problems and perspectives.

Nader Masmoudi

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MR3157048 35-02; 35Q20, 35Q70, 82B40, 82C22

Gallagher, Isabelle; Saint-Raymond, Laure; Texier, Benjamin

From Newton to Boltzmann: hard spheres and short-range potentials. (English)

Zurich Lectures in Advanced Mathematics.

European Mathematical Society (EMS), *Zürich*, 2013, xii+137 pp.,
ISBN 978-3-03719-129-3

In this monograph, the authors are concerned with the rigorous derivation of the (irreversible) classical Boltzmann equation as a limiting case of (reversible) Newtonian molecular dynamics. They provide a self-contained re-visitation of the argument outlined by O. E. Lanford III in his seminal work [in *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)*, 1–111. Lecture Notes in Phys., 38, Springer, Berlin, 1975; MR0479206] for the local-in-time validity of the Boltzmann equation for hard spheres in the Boltzmann-Grad limit of the Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. Moreover, the authors reconsider the results of F. G. King's thesis [*BBGKY hierarchy for positive potentials*, Univ. California, Berkeley, 1975; MR2625983], which extend Lanford's argument to a Hamiltonian system of particles with short range, repulsive potentials.

In essence, following Lanford and King, the monograph shows that for hard spheres and Hamiltonian systems with short range repulsive interactions, and for short times, statistical states solving the BBGKY hierarchy converge, in the Boltzmann-Grad limit, to solutions of the so-called Boltzmann hierarchy. In particular, independent initial states of the BBGKY hierarchy result, in the limit, into factorized solutions of the Boltzmann hierarchy (propagation of molecular chaos), each factor solving the Boltzmann equation. To prove the convergence, the authors apply Lanford's strategy to compare a suitable series expansion of the BBGKY solutions (in terms of sums of collision trees) with the corresponding one for Boltzmann hierarchy, by checking the term by term convergence.

The book has the merit that it provides nontrivial missing details of Lanford's argument, in particular those concerning the term-wise convergence, incompletely presented in the previous literature (even in the hard-sphere case), and that it clarifies some obscure points and fills some gaps of King's work (mostly related to the proof of the term-wise convergence).

The monograph is divided into four major parts (15 chapters). The first part is a selective, contextual introduction to the main problems and results of the book.

Part II is concerned with the hard-sphere case. It provides a rigorous derivation of the BBGKY hierarchy and a precise statement for the convergence of solutions of BBGKY hierarchy (Theorem 8). This part also includes important consideration on independence, propagation of molecular chaos, and an insight into the strategy of the convergence proof.

Part III deals with short range repulsive potentials. Here, the derivation of the BBGKY hierarchy is more delicate. The convergence theorem (Theorem 11) is formulated for smooth, compactly supported, nondecreasing, repulsive potentials, singular at the origin (Assumption 1.2.1), which satisfy a condition (8.3.1) ensuring that the scattering cross-section is well defined.

The principal contribution of the book emerges from Part IV, which concludes the proofs of the main convergence results (Theorems 8 and 11). The arguments are similar, both for hard spheres and short range potentials, regardless of the nature of the interactions. The key point of the analysis is the proof of the term-wise convergence. This is based on the elimination (control) of re-collisions (which are "bad" events leading to an evolution different from the Boltzmann behavior). To control re-collisions, the authors apply explicitly the properties of the scattering cross-section.

The monograph includes a list of references and a notation index.

After the publication of the printed book, the authors provided an on-line erratum to Chapter 5 at `www.ems-ph.org/books/173/Erratum-chapter5.pdf` to remove some inconsistencies relative to the functional framework of the book.

The monograph is a good reference for researchers and graduate students interested in the fields of mathematical physics and partial differential equations.

An alternative approach (and also extension to stable short range potentials) has been recently published [M. Pulvirenti, C. Saffirio and S. Simonella, Rev. Math. Phys. **26** (2014), no. 2, 1450001; MR3190204].

Cecil Pompiliu Grünfeld

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