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Complex Contour Integral Representation of Cardinal Spline Functions

Walter Schempp

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To my Parents

The mathematics that we use
in spline approximation theory
is Eulerian in character.

I.J. Schoenberg, 1973

Foreword

The simplest and most versatile functions are the continuous piecewise linear functions. Their undesirable corners are eliminated by $m-1$ successive integrations, the result being a spline function $s(x)$ of degree m . In other words: While the m th derivative of a polynomial of degree m is a constant, the m th derivative $s^{(m)}(x)$ of a spline function is a piecewise constant function. Its discontinuities are the knots of $s(x)$.

Besides the m constants of integration obtained by passing from $s^{(m)}(x)$ to $s(x)$, the spline $s(x)$ depends also on the jumps of $s^{(m)}(x)$, not to mention the non-linear parameters giving the location of the knots.

An important property of the class of splines $s(x)$ of degree m with fixed knots is its plasticity: By this I mean that its shape is a local matter; this is due to the fact that a spline of least support, a so-called B-spline, has precisely $m+2$ consecutive knots, and that all splines are unique linear combinations of consecutive B-splines to which we add an arbitrary polynomial of degree $m-1$.

Spline functions thus appear as a natural generalization of polynomials. As such they have considerably enriched the Theory of Approximations. A fundamental result of this theory is the Weierstrass polynomial approximation theorem. However, a polynomial of very high degree can not possibly be useful from the numerical point of view. Now spline functions of degree 2, 3, 4, or 5, allow us to obtain acceptable approximations to complicated functions, provided that their knots are judiciously chosen. Even the Bernstein polynomials may be replaced by the Bernstein splines having similar variation diminishing properties as well as improved approximation properties.

The simplest spline functions are those having as knots a biinfinite sequence of points in arithmetic progression. These are the so-called Cardinal Spline Functions. All results on Cardinal Spline Interpolation have so far been obtained by the Eulerian methods of generating functions and difference equations.

It is most desirable that these problems are now studied by Professor Walter Schempp by the use of the powerful methods of complex contour integral representations and Integral Transforms. This more sophisticated approach promises farreaching developments.

Mathematics Research Center
University of Wisconsin - Madison

December 1980

I.J. Schoenberg

Preface

The complex integral representation of cardinal splines by means of linear integral transform methods is a very effective tool for dealing with these functions. In particular, their asymptotic behaviour as the degree tends to infinity can be analyzed conveniently and handled adequately by integral transform methods.

This is *not* a treatise on the theory of cardinal spline functions. It is a set of lecture notes aimed at acquainting the student with the complex contour integral representation approach to cardinal spline functions, to wit, the author's personal approach to the subject [38], [39]. The basic idea is to use a suitable inverse integral transform instead of the direct transform itself and then to have recourse to the methods of complex analysis. Special emphasis is placed on the cardinal exponential splines, which are represented by an application of the inverse bilateral Laplace transform, and on the cardinal logarithmic splines, the complex contour integral representation of which is obtained by the inverse unilateral Mellin transform. It is well known that the convergence properties of these two kinds of splines are totally different. Nevertheless, the method of complex contour integral representation yields a unified treatment of both cases and gives powerful insight into what actually happens. In particular, the Newman-Schoenberg phenomenon loses some of its mystery.

Besides presenting an outline of inverse integral transform technique, we study several closely related topics. These include 1) various complex integral representations of the basis spline functions, 2) a useful complex contour integral representation of the Euler-Frobenius polynomials and its consequences, and 3) the classical Méray-Runge phenomenon (as preparation for Newman-Schoenberg).

It is our hope that these notes will be useful to a broad audience, interested in present developments of approximation theory.

Mathematicians specialized in the field of spline approximation are sometimes unfamiliar with the methods of integral transform analysis. For this reason we give full details of the necessary tools from this important branch of applied mathematics, with the intention of presenting the material in a self-contained manner. Furthermore, we feel that our approach to cardinal spline functions provides a very instructive illustration of the applications of inverse integral transform techniques combined with complex variables methods to recent problems arising in approximation theory.

Each section ends with a few references and comments. Basically we have chosen only those references which we feel are most useful from our point of view. Nonetheless, we refer the reader to Max Weber's aphorism: "Das Wichtigste steht natürlich in den Anmerkungen."

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W.S.

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