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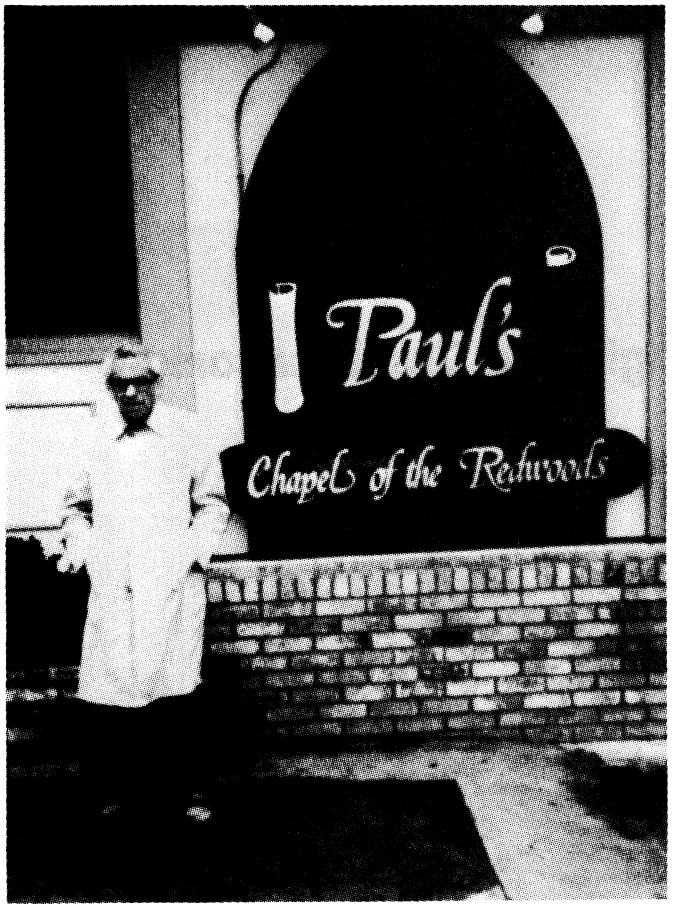
Logic and Combinatorics



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Professor Erdos next to an Arcata landmark.

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Stephen G. Simpson, Editor

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PREFACE

This volume is the proceedings of the AMS-IMS-SIAM Joint Summer Research Conference "Applications of Mathematical Logic to Finite Combinatorics." The conference took place August 4 - 10, 1985 at Humboldt State University in Arcata, California.

* * *

Recent years have witnessed the appearance of several remarkable results to the effect that certain mathematically natural theorems of finite combinatorics (often Ramsey-type theorems) are unprovable in certain familiar logical systems (usually systems of first- or second-order arithmetic or of set theory). These developments have stimulated several lines of research. On the combinatorial side, there has been renewed interest in Ramsey theory and in allied topics such as well quasiordering theory. On the logical side, there have been new developments in nonstandard models of arithmetic and set theory, and in proof theory.

The purpose of the conference was to bring together people who are interested in these matters. Of course there were many talks dealing with unprovability results and "fast-growing functions" in finite combinatorics. But more generally, the conference provided a forum for logicians and combinatorists to discuss a broad range of topics which are of mutual interest.

* * *

The conference was a great success! There was a truly exceptional turnout of mathematicians representing many fields including Ramsey theory, well quasiordering theory, function hierarchies, combinatorial set theory, ordinal notations, computational complexity, and nonstandard

models of arithmetic and set theory. Among the more than seventy participants were many prominent figures including Ajtai, Baumgartner, Blass, Buchholz, Buss, Carlson, Clote, Compton, Deuber, Erdős, Fleissner, Frankl, Graham, Hajnal, Harrington, Hindman, Jockusch, Kanamori, Kierstead, Larson, Laver, Leeb, Marker, Mathias, Miller, Moh, Moran, Müller, Mycielski, Owings, Pearce, Pouzet, Prömel, Pudlák, Ressayre, Rival, Robertson, Rödl, Rothschild, Schmerl, Shelah, Sipser, Smoryński, Spencer, Szemerédi, Takeuti, Todorčevic, Trotter, Voigt, Wainer, and Yao. The talks were exciting and the informal discussions even more so. The attractive campus and excursions to the redwood forests helped to make the conference as enjoyable as it was productive.

* * *

The papers in this volume are a fairly representative and diverse collection. Nevertheless, some central themes stand out.

A large number of the papers deal with one or another aspect of "unprovable theorems and fast-growing functions." My own paper of that title is an introductory survey which should help to place some of the other papers into an appropriate context of recent related research. The Buchholz-Wainer paper presents a novel, elegant proof of the often-stated but seldom-proved result relating the fast-growing function hierarchy to provability in first-order Peano arithmetic, PA. This result is then used to establish the unprovability in PA of Goodstein's Theorem on integers written in exponential notation. Expanding on the latter topic, the papers of Abrusci and Abrusci-Girard-van de Wiele present a category-theoretic approach to generalized Goodstein sequences.

The papers of Brackin and Blass-Hirst-Simpson include axiomatic studies of various well known combinatorial theorems. The contributions by Loebl-Matoušek, Nešetřil-Thomas, and Okada-Takeuti elucidate the relationship between finite trees, ordinal notations, and unprovability results. A major paper by Ressayre uses nonstandard models to obtain finite combinatorial theorems which are "true" but not provable in Zermelo-Fraenkel set theory.

Coloring theorems, of which Ramsey's Theorem is the prototype, were much discussed at the conference. Bergelson's paper presents several new results in the dynamical systems approach to this subject. Axiomatic aspects of the dynamical approach are discussed in the Blass-Hirst-Simpson

contribution. Two papers, by Blass and Hindman respectively, discuss ultrafilters related to the famous theorem of Hindman on infinite sets of integers all of whose finite sums have the same color. Aspects of the related theorems of van der Waerden and Szemeredi are discussed in the contributions by Bergelson and Brackin.

Another subject which was extensively aired at the conference is well quasiordering theory. This topic is touched on in the papers by Loebl-Matoušek, Nešetřil-Thomas, Okada-Takeuti and Simpson, which have already been mentioned. A major contribution to the theory is the van Engelen-Miller-Steel paper, which includes, among other things, a simplified treatment of the "minimal bad array" lemma.

Combinatorial set theory was also much discussed and is well represented in this volume by the papers of Ackman-Owings, Baumgartner-Hajnal, and Erdős.

An important paper of Pudlák presents upper and lower bounds for what might be called the ultrafinitist form of Gödel's Second Incompleteness Theorem.

A last-minute addition to this volume is the remarkable paper of Friedman, Robertson and Seymour. The authors show that strong set existence axioms are needed to prove one of the most important theorems of graph theory, namely the well quasiordering of the class of finite graphs under minor embeddability.

* * *

I would like to thank the Joint Summer Research Conference Committee, chaired by R. O. Wells, Jr., for inviting me to organize this conference. I would also like to thank the other members of the organizing committee (Harvey Friedman, Ron Graham, Leo Harrington, and Jeff Paris), and also Angus Macintyre, for their help. Thanks are due also to the National Science Foundation, which provided major financial support, and to the Association for Symbolic Logic, which cosponsored the conference. My deepest gratitude is reserved for the people at the American Mathematical Society and especially Carole Kohanski, who cheerfully and flawlessly handled all of the more burdensome details of organizing and running this conference.

Stephen G. Simpson

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