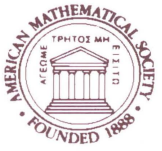
CONTEMPORARY MATHEMATICS

98

Every Planar Map is Four Colorable

Kenneth Appel and
Wolfgang Haken



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Kenneth Appel and
Wolfgang Haken

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Authors' Note

In this volume we present an emended version of our proof of the Four-Color Theorem in a form as self-contained as we can make it. In addition, we have included a proof that four coloring of planar maps can be done in polynomial time. It had been suggested to us in 1976 that this result was an obvious consequence of the procedure of replacing reducible configurations by subgraphs with fewer vertices. A careful examination of the argument for situations in which reducible configurations are immersed rather than embedded in our triangulations showed a need for a rather more sophisticated argument. Since Section 3 of Part II was somewhat sketchy for careful readers, an appendix to Part II is provided to present "immersion reducibility" in detail and to prove the polynomial time result.

We begin with an introduction that is largely intended for the non-specialist. The remainder of the volume consists, in addition to the appendix to Part II, of emended copies of the two papers "Every planar map is four colorable Part I: Discharging and Part II: Reducibility" originally published in the *Illinois Journal of Mathematics* as well as the detailed supplements to these papers which were circulated with the *Illinois Journal* in microfiche form.

Although the papers themselves have a few pages of introductory material, the introduction to the volume is itself intended to be self-contained, and includes some information mentioned in the papers.

The introduction consists of five sections. Section 1 is an introduction to the history of the Four-Color Problem and an outline of the techniques used in the proof. It consists of material that appeared in [5]. (References here are to the bibliography of the introduction.) A reader familiar with the problem may wish to skip Section 1. Section 2 is an introduction to D- and C-reducibility. It expands upon the brief discussion of reducibility in Section 1. A formal presentation of D- and C-reducibility is given in Sections a, . . . , k of the appendix to Part II. Section 3 provides more detail on unavoidable sets and our discharging procedure. It includes a detailed discussion of the "error correction routine" which is typical for this type of proof. This material appeared in [6]. Section 4 describes the organization of the proof in the two papers [3] and [4], a supplement, and a supplement to the supplement. It also describes some of the work done by other investigators in checking the proof or providing parallel verification. Section 5 describes our own checking procedures, which used both careful hand checks and computer verifications of work originally done by hand. This introduction has a comparatively short bibliography. Since [3] (as corrected) is part of

this book, many of the references in the introduction are made to the bibliography of [3]. A reference to article number n in the bibliography of [3] is given as [3,n]. The remainder of the book consists of the two papers [3] and [4], the appendix to [4], and the supplements.

Each page of this book that is reprinted from an earlier publication has two numbers — a page number for this book and the original page number. The new material, including the introduction, has single page numbers. References in the contents are to the original page number only. The number is prefaced by "D" for [3] the discharging paper, "R" for [4] the reducibility paper, "S" for the supplement, and "C" for the checklists (the supplement to the supplement).

It is important to understand how the supplement and checklists are related to the exposition of the proof in [3]. In the sense of mathematical papers, [3] is self-contained, since trivial details are usually left to the reader rather than clogging the exposition. Here, however, this approach leads to some difficulty. For example, on page D-462, the first line of the proof of the $q_{TS}(V_5)$-Lemma states "This is proved by straightforward enumeration of all possible cases of $q_{TS}(V_5) > 0$." The reader can then discover that the case analysis for this enumeration takes 29 pages in the supplement, and that the details to verify these cases require 107 pages of the corresponding checklists. We felt that these should be supplied as a courtesy to the reader although admittedly it is somewhat unusual to provide such detail. The tables of contents for the supplement and checklists are displayed in matching columns so that the material in the checklists corresponding to the analysis in the supplement appears on the same line. While it is possible to find many of the lemmas in the table of contents of the supplement, not all of them are there. Thus, to aid the careful checker, a separate table of lemmas gives the page on which each lemma is stated and where its proof starts.

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