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Deformation Theory and Quantum Groups with Applications to Mathematical Physics

Proceedings of an AMS-IMS-SIAM
1990 Joint Summer Research Conference
held June 14–20
at the University of Massachusetts, Amherst
with support from the National Science Foundation

Murray Gerstenhaber
Jim Stasheff
Editors

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Preface

Drinfel′d's 1986 contribution to the 1986 International Congress of Mathematicians in Berkeley focused the attention of the mathematical world on "quantum groups". Drinfel′d observed that certain structures playing a central role in the statistical and wave mechanics of Baxter and Yang were in fact Hopf algebras, and that those appearing were of a kind not previously studied but "deformations" of certain classical ones. Quantization had produced these structures (or perhaps deformation had produced the quantization). Although "quantization" may refer to various processes arising in physical quantum theory, some can be singled out as essentially algebraic and closely related to the deformation theory of algebras (commutative, Lie, Hopf, etc.). (The obeisance to physics is sometimes indicated by denoting the deformation parameter by the symbol for Planck's constant, $\hbar$ or by "q" which is usually interpreted as $\exp i\hbar$.)

The mathematical origins of quantization are venerable, but the new subject of quantum groups is growing so fast, with interactions amongst so many branches of mathematics and physics (including, e.g., knot theory and invariants of 3-manifolds), that there is no clear overview, nor can we expect one for some years. The purpose of the AMS-IMS-SIAM Joint Summer Research Conference in 1990, from which this volume stems, was to bring together researchers both in some areas recently opened and in ones such as "q special functions" which had their origins in the last century but whose relevance to modern physics has only recently been understood. At best, the conference which we organized in June of 1990, while bringing together an international gathering of many mathematicians and a few physicists, could only provide a "snapshot" of progress in a few areas. It almost completely omitted, for example, applications to topology. While we regret that most of the invited Soviet researchers could not attend, this was partly because of a happy fact: peristroika had opened their community and many were then preparing for the Workshop on Quantum Groups, Deformation Theory, and Representation Theory held at the new Euler International Mathematical Institute, St. Petersburg (the still Leningrad) that same October! (This partly erased the unhappy memory that Drinfel′d had not been present at Berkeley in 1986;

his contribution was read by Pierre Cartier. Happy indeed are the memories from the ICM in Kyoto later in the summer of 1990 when Drinfel′d received the Fields Medal for his work, especially that on quantum groups.)

As for the subject matter, one of the oldest forms of algebraic quantization amounts to the study of deformations of a commutative algebra A (of classical observables) to a noncommutative algebra A_h (of operators) with the infinitesimal deformation given by a Poisson bracket on the original algebra A. Physics provides many examples. Perhaps the oldest is Moyal's, of which the more modern generalization is the quantization (deformation) of the C^∞ smooth functions on a symplectic manifold. Deformations have been studied intensively from this point of view by Lichnerowicz and his school, as noted in extensive references by Drinfel′d in his ICM talk. In the past decade, a new source of examples has come from the physics of completely integrable systems (KdV, KP, bi-hamiltonian systems) and from the inverse scattering method. As in the original work of Moyal, statistical mechanics and quantum field theory have both called attention to a single new mathematical structure, in this case, the Yang-Baxter equations. The intimate relation between these equations and "quantum groups" is well described in the language of deformation theory.

Quantum groups are not groups at all, but special kinds of Hopf algebras of which the most important are closely related to Lie groups. Even though it is not obvious from their definitions, all those of physical interest seem to be deformations of either the Hopf algebra $F(G)$ of functions on a Lie group G or of a universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The latter can be regarded as the Hopf algebra of distributions-with-support-at-the-identity of G where $\mathfrak{g}$ is the Lie algebra of the group G.

From the point of view of noncommutative geometry, as well as from that of physical observables, it is the multiplication in $F(G)$ (but not its comultiplication) or the comultiplication in $U(\mathfrak{g})$ (but not its multiplication) which should be deformed. For reductive G, it is always possible to deform the one while preserving the other (and every deformation is equivalent to one which does this) even though the formulas in the literature usually deform both. That such "preferred deformations" are possible is a consequence of the appropriate cohomology theory.

We have not tried to arrange the contents by subject. We would surely make too many mistakes. Worse, it might be an invitation to the reader to pick and choose when browsing is more appropriate. Some of the articles here may be obsolete before they are in print. For that we make no apology. In fact, we hope it will be the case, since rapid obsolescence is a measure of progress, which in the subjects treated here seems to be marching with giant strides.

Murray Gerstenhaber
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Murray Gerstenhaber and Jim Stasheff, Editors

Quantum groups are not groups at all, but special kinds of Hopf algebras of which the most important are closely related to Lie groups and play a central role in the statistical and wave mechanics of Baxter and Yang. Those occurring physically can be studied as essentially algebraic and closely related to the deformation theory of algebras (commutative, Lie, Hopf, and so on). One of the oldest forms of algebraic quantization amounts to the study of deformations of a commutative algebra A (of classical observables) to a noncommutative algebra A_h (of operators) with the infinitesimal deformation given by a Poisson bracket on the original algebra A.

This volume grew out of an AMS-IMS-SIAM Joint Summer Research Conference, held in June 1990 at the University of Massachusetts at Amherst. The conference brought together leading researchers in the several areas mentioned and in areas such as "q special functions", which have their origins in the last century but whose relevance to modern physics has only recently been understood. Among the advances taking place during the conference was Majid's reconstruction theorem for Drinfel′d's quasi-Hopf algebras. Readers will appreciate this snapshot of some of the latest developments in the mathematics of quantum groups and deformation theory.

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