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The Reconstruction of Trees from Their Automorphism Groups

Matatyahu Rubin

American Mathematical Society

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Summary

This work addresses the following question: "Find meaningful necessary and sufficient conditions for two trees to have isomorphic automorphism groups." To answer this question, we consider a more general type of structures, called unary trees.

Definition: (a) A poset $\langle T, < \rangle$ is a *tree*, if for every $t \in T$, $\{s \in T \mid s < t\}$ is linearly ordered by $<$. $M = \langle T, <, \{P_i\}_{i \in I} \rangle$ is a *unary tree* (*U-tree*), if $\langle T, < \rangle$ is a tree, and for every $i \in I$, $P_i \subseteq T$. Informally, A U-tree is a structure which is a tree together with a list of named subsets, each of which is required to be invariant under all automorphisms of the structure. So, for the above M, $Aut(M) = \{f \in Aut(\langle T, < \rangle) \mid (\forall i \in I)(f(P_i) = P_i)\}$.
(b) Let M be as in (a), then $T(M)$ denotes $\langle T, <, \sim \rangle$, where $s \sim t$ means that there is $g \in Aut(M)$ such that $g(s) = t$. That is, we replace the named subsets of M by the single binary relation $\sim$.
(c) A class L of U-trees is *faithful*, if for every $M, N \in L$: if $Aut(M) \cong Aut(N)$, then $T(M) \cong T(N)$. Clearly, for trees: $T(M) \cong T(N) \Rightarrow M \cong N$. So, for a faithful class L of trees and $M, N \in L$: $Aut(M) \cong Aut(N) \Rightarrow M \cong N$.

The answer to the main question cannot be easily stated. One ingredient in the answer, is finding large faithful classes of U-trees. Theorem 1 below is an example of such a result. Our more general theorems come close to showing that the faithful class of theorem 1 cannot be enlarged.

Definition: Let $M = \langle T, <, \ldots \rangle$ be a U-tree. **(a)** $Max(M)$ denotes the set of maximal elements of M. $A \subseteq T$ is an *interval* of M, if A is linearly ordered by $<$, and for every $s, t \in A$ and $u \in T$: if $s < u < t$, then $u \in A$. Let $s, t \in T$. $Or(t; s)$ denotes $\{f(t) \mid f \in Aut(M)$ and $f(s) = s\}$. t is called a *successor* of s ($t \in Suc(s)$), if $s < t$ and $\{s, t\}$ is an interval.
(b) M is *complete*, if: (1) Every $\emptyset \neq A \subseteq T$ has an infimum ($\inf(A)$); (2) Every interval in T has a supremum; and (3) If A and B are disjoint nonempty intervals, then $\inf(A) \neq \inf(B)$.

Every U-tree M has a naturally defined completion N. The named subsets of N are: M, and all the named subsets of M. It follows that $Aut(N) \cong Aut(M)$. So, when seeking faithful classes, we may consider only complete U-trees.

Theorem 1: The class of all U-trees M satisfying conditions (1)-(4) below, is faithful. (1) M is complete. (2) For every $s \in M$: $|Suc(s)| \neq 1$. (3) For every $s \in M$: either for all $u, v \in Suc(s)$: $u \sim v$, or for all distinct $u, v \in Suc(s)$: $u \not\sim v$. (4) For every $s \in M$ and $t > s$: if $|Or(t; s)| \leq 2$, then $t \in Suc(s) - Max(M)$.

For the class of $\aleph_0$-categorical U-trees, the main question has the following complete answer. A class K_{CAT} of $\aleph_0$-categorical U-trees is defined by listing five properties similar to properties (1)-(4) of theorem 1. (See 0.6.) We prove that K_{CAT} is faithful, and that if M is an $\aleph_0$-categorical U-tree, then there is $N \in K_{CAT}$ such that $Aut(M) \cong Aut(N)$.
The situation in the class of all trees is similar, but more complex.

The Reconstruction of Trees from Their Automorphism Groups

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Trees, sometimes called semilinear orders, are partially ordered sets in which every initial segment determined by an element is linearly ordered. This book focuses on automorphism groups of trees, providing a nearly complete analysis of when two trees have isomorphic automorphism groups. Special attention is paid to the class of $\aleph_0$-categorical trees, and for this class the analysis is complete. Various open problems, mostly in permutation group theory and in model theory, are discussed, and a number of research directions are indicated. Aimed at graduate students and researchers in model theory and permutation group theory, this self-contained book will bring readers to the forefront of research on this topic.

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