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Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups

AMS-IMS-SIAM Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups
June 13–19, 1992
Mount Holyoke College

Paul J. Sally, Jr.

Moshe Flato
James Lepowsky
Nicolai Reshetikhin
Gregg J. Zuckerman
Editors

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PREFACE

In July 1982, the 14th AMS-SIAM Summer Seminar on *Applications of Group Theory in Physics and Mathematical Physics* was held in Chicago. A volume of papers submitted by speakers at the Seminar was published by the AMS in 1985 under the same title as Volume 21 of the Series *Lectures in Applied Mathematics.* This volume was edited by three of the editors of the present volume (M. Flato, P. Sally and G. Zuckerman). The aim of the Chicago meeting was "to bring together a broad spectrum of scientists from theoretical physics, mathematical physics and various branches of pure and applied mathematics in order to promote interaction and an exchange of ideas and results of common interest".

At that time, the idea may have appeared to be somewhat revolutionary. Eight years later, it was obvious to everyone that this attitude had become part of mainstream mathematics, to such an extent that three of the four 1990 Fields medalists were inspired by such an interaction.

As a result, the editors of the Seminar volume decided that it was time for an update on the 1982 conference. Thus was organized (with J. Lepowsky and N. Reshetikhin added to the Organizing Committee) the June 13-19, 1992 AMS-IMS-SIAM Joint Summer Research Conference at Mount Holyoke College, entitled "Conformal Field Theory, Topological Field Theory and Quantum Groups". The present volume contains papers submitted by speakers at the Conference. The papers do not always coincide 100% with the actual lectures, but in most cases are close to the lectures, and they always deal with matter that was the subject of the Conference. Some of the speakers have not yet submitted papers (for a variety of reasons) but we feel that this volume should be published without further delay.

The Conference turned out to be much more mathematically oriented than the previous Seminar, reflecting the fact that many of the areas of theoretical physics which were discussed in the Seminar became leading areas in mathematics in the ten years between the meetings. (One of the subjects treated at the Conference, Quantum Groups, was barely born at the time of the Seminar). We therefore added the words "Mathematical Aspects" to the title to reflect that situation. Simultaneously, the partition of the contributions into well-defined subjects became almost impossible because several mathematical methods are common to two or more physical applications. We shall thus present the contributions as a continuous list.

The first group of papers deals with one of the aspects of Conformal Field Theory (CFT), the so-called Vertex Operator Algebras (VOA) or superalgebras (SVOA) and their representations:

1. **Y.-Z. Huang and J. Lepowsky** reformulate the notion of VOA, interpreted geometrically by one of the authors (YZH) using certain moduli spaces of spheres with punctures and local coordinates, in terms of structures called "operads," which first arose in the homotopy-theoretic characterization of iterated loop spaces. Doing so, they show that the rich geometric structure of CFT and the rich algebraic structure of VOA share a precise common foundation in basic operations associated with a certain two-dimensional object. This paper presents a step in an approach which was just beginning around the time of the Seminar and has developed into a full-fledged theory with numerous ramifications.

2. **C. Dong** deals with the representations of the moonshine module VOA, whose automorphism group is the Monster finite simple group. He proves the significant result that the adjoint representation is the unique irreducible one (part of a conjecture of Frenkel-Lepowsky-Meurman) and that every representation is a multiple of it.

3. **C. Dong and G. Mason** present a new (and rigorous) construction of the moonshine module VOA as a $\mathbb{Z}_3$-orbifold structure in the sense of CFT, and more generally, they discuss its construction as a $\mathbb{Z}_p$-orbifold structure. (The construction of Frenkel-Lepowsky-Meurman was as a $\mathbb{Z}_2$-orbifold structure.) The main theorem provides a nontrivial instance of "mirror symmetry". This work includes a characterization of this VOA as an irreducible module for the affinization of the Griess algebra.

4. **V. Kac and W. Wang**. Recently, Zhu has constructed an associative algebra $A(V)$ corresponding to a VOA V such that up to equivalence, the irreducible representations of V and $A(V)$ are in one-to-one correspondence. In addition, to any V-module M, Frenkel and Zhu have associated an $A(V)$-module $A(M)$ in terms of which the fusion rules can be determined. An advantage of these constructions is that $A(V)$ and $A(M)$ can usually be computed explicitly. This was used by several authors (e.g., Wang, and Dong, Mason and Zhu) to prove rationality and calculate fusion rules for series of representations of the Virasoro algebra. In this paper, the authors extend these results to superalgebras (SVOA's) and concentrate on those of Neveu-Schwarz type ($N = 1$). In particular they prove rationality and study fusion rules for SVOA's corresponding to an affine Kac-Moody superalgebra introduced by Kac and Todorov in relation to superconformal current algebras. They also show rationality for the SVOA's generated by charged and neutral free fermionic fields. This paper represents an extension of the VOA theory to superalgebras, needed in several physical applications.

A central object in CFT is the Khnizhnik-Zamolodchikov (KZ) equation, which has been extensively studied since it was introduced in 1984 in relation to current algebra Wess-Zumino-Novikov-Witten models. Here:

5. **Y. Stanev and I.T. Todorov** review the classification (that they obtained in collaboration with L. Michel) of all polynomial solutions of the KZ equation for the $SU(1,1)$-invariant 4-point amplitude of a primary field in $\mathfrak{su}(2)$ conformal current algebra.

The theory of knots in 3-manifolds is related to all the subjects dealt with at the Conference, including CFT, the Topological Field Theory (TFT) of Witten and

Quantum Groups at roots of unity. Here:

6. T. Kohno gives a lower bound for a certain topological invariant of a knot, the tunnel number, in terms of the Jones-Witten invariants for a knot in a closed oriented 3-manifold. In particular for the manifold S^3 and Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ he shows that the lower bound is described by special values of the Jones polynomial.

TFT arose from the Chern-Simons gauge theory, the classical phase space of which can be viewed as the moduli space $\bar{S}_g$ of flat connections on the trivial principal G-bundle (G a compact simple Lie group) on a compact oriented 2-manifold of genus g. $\bar{S}_g$ contains an open dense subset S_g that is a symplectic manifold and is conjectured to be related to invariants of 3-manifolds. Here:

7. L. Jeffrey and J. Weitsman summarize some of their recent work on the symplectic geometry of $\bar{S}_g$, for $G = SU(2)$. They use the existence of Hamiltonian torus actions in these $\bar{S}_g$ and the images of the corresponding moment maps to find a simple description of $\bar{S}_g$, and they use this to compute such quantities as symplectic volumes.

A concept of tensor product of representations is suggested by fusion rules in CFT, and tensor products appear in many ways (arising from coproducts) in Quantum Groups. Computing and understanding suitably-defined tensor product multiplicities is therefore an important theme in both subjects. Here:

8. G. Georgiev and O. Mathieu compute some modified tensor product multiplicities for indecomposable tilting modules of Chevalley groups (over a field of characteristic p large enough), showing that they are the same as those obtained (from the recently-studied conformal-field-theoretic tensor product) for an associated complex affine Kac-Moody algebra. They conjecture that this coincidence can be explained through a lifting to quantum groups. This tensor category also turns out to be the same as one constructed recently by S. Gelfand and D. Kazhdan that was discussed by Kazhdan at the Conference. (The talks of Huang and Lepowsky at the Conference also presented the construction and properties of a tensor category of representations of a VOA.)

Another paper touching both CFT and Quantum Groups is the following:

9. K. Aomoto and Y. Kato deal with some mathematical problems around the KZ equation in 2-dimensional CFT and Quantum Groups. Considering a holonomic system of q-difference equations, they construct special q-cycles ($q = e^{2\pi i\tau}, \mathrm{Im}(\tau) > 0$) for n-dimensional Jackson integrals on which the canonical action of the permutation group gives a linear representation satisfying the Yang-Baxter equation.

The remaining four papers deal mostly with the "quantum groups" part of the Conference:

10. M. Flato and D. Sternheimer review the deformation-quantization approach both in quantum mechanics and quantum field theory, i.e., the star-product theory that can now be viewed as a predecessor of quantum groups. They also review the related notion of star-representation of Lie groups and the case of closed star-products, classified by cyclic cohomology and related to noncommutative geometry. Finally they explain the relations with quantum groups, showing in what sense quantum groups are special cases of star-product algebras.

11. M.A. Semenov-Tian-Shansky describes various notions of double for a quantum group in the FRT (Faddeev-Reshetikhin-Takhtajan) approach (with R-matrices supposed known in advance) and concentrates on what is called the Heisenberg double, the quantum analogue of the Hopf algebra of functions on T^*G (the cotangent bundle on a Poisson Lie group G), and its twisted versions (nontrivial deformations associated with outer automorphisms of the underlying algebras). The quantum duality principle (a pairing between the quantum Hopf algebras of functions on G and G^*) then permits him to define a quantum Fourier transform.

12. C. Frønsdal and A. Galindo base their study on an abstraction (definition in the structure and not in a representation) of the T-matrices of integrable models, that they call "the universal T-matrix" and is the Hopf algebra dual form expressed in terms of generators. They start with the Woronowicz quantum group $\mathcal{U}_{q,q'}(\mathfrak{gl}(2))$ to make their formalism more transparent, and get its (solvable) bialgebra dual. For $\mathfrak{gl}(3)$ they then discover a new (esoteric) quantum deformation, not of the coboundary type (and with nonsolvable dual) but for which a Yang-Baxter R-matrix still exists. They then extend these results to $\mathfrak{gl}(n)$, obtain also the corresponding doubles and end with a few remarks on multiparameter deformations.

13. V. Ginzburg, N. Reshetikhin and E. Vasserot obtain an analogue of the Weyl correspondence (between irreducible representations of the symmetric groups and certain finite-dimensional representations of the general linear groups, in the complex case) in the quantum affine setup, and provide its geometric interpretation. The q-analogue of the symmetric group algebra is a Hecke algebra, and the q-Weyl correspondence was described already by Jimbo and interpreted geometrically by Lusztig and coworkers. Here the authors extend the formalism to the affine case using a notion of polynomial tensor representation which they interpret geometrically using affine flag varieties.

Finally, we want to express our thanks to the American Mathematical Society for its excellent logistical support, and especially to Carol Kohanski, who helped in the organization at Mount Holyoke, and to Donna Harmon, who helped in collecting the contributions. Last but not least we thank all the participants, speakers with written contributions, speakers without written contributions, and all those who attended, for the very fruitful atmosphere that was created at Mount Holyoke College during the Conference.

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Paul Sally

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This book contains papers presented by speakers at the AMS-IMS-SIAM Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, held at Mount Holyoke College in June 1992. One group of papers deals with one aspect of conformal field theory, namely, vertex operator algebras or superalgebras and their representations. Another group deals with various aspects of quantum groups. Other topics covered include the theory of knots in three-manifolds, symplectic geometry, and tensor products. This book provides an excellent view of some of the latest developments in this growing field of research.

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