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Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes

Leovigildo Alonso Tarrío
Ana Jeremías López
Joseph Lipman

American Mathematical Society

http://dx.doi.org/10.1090/conm/244

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1991 *Mathematics Subject Classification.* Primary 14F99;
Secondary 13D99, 14B15, 32C37.

Library of Congress Cataloging-in-Publication Data

Alonso Tarrío, Leovigildo, 1962–
Studies in duality on noetherian formal schemes and non-noetherian ordinary schemes / Leovigildo Alonso Tarrío, Ana Jeremías López, Joseph Lipman.
p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 244)
Includes bibliographical references and index.
ISBN 0-8218-1942-9 (alk. paper)
1. Duality theory (Mathematics) 2. Schemes (Algebraic geometry) I. Jeremías López, Ana, 1962– . II. Lipman, Joseph. III. Title. IV. Series: Contemporary mathematics (American Mathematical Society) ; v. 244.
QA564.A52 1999
515′.782–dc21 99-42685
CIP

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Preface

This volume comprises three essentially independent, but related, papers treating the foundations of Grothendieck Duality on noetherian formal schemes and on not-necessarily noetherian ordinary schemes. Here, briefly, is what is done and what is left undone.

Grothendieck Duality starts with the existence of a right adjoint for the (suitably restricted) derived direct image functor associated to a proper map, and the compatibility of such a right adjoint with flat base change. Our treatment in the first paper is the first for arbitrary noetherian formal schemes. (The classical case of ordinary schemes is the one where the topology on the structure sheaf is discrete.) It is indicated how the main results synthesize several duality-related topics such as local duality, formal duality, dualizing complexes, and residue theorems. The overall approach is abstract, in the style of Verdier and Deligne. Enlivening concrete interpretations—often involving differential forms—are left to another occasion.

It should be noted that the proof of the base-change theorem on formal schemes given in §7 of the first paper uses the special case of base-change on ordinary schemes, under weaker assumptions than those which support published proofs of the latter. There is at least outlined in the third paper a method for proving a sufficiently general base change theorem on ordinary schemes, even without noetherian hypotheses. Moreover, while details are not given, it seems that the method could be modified so as to apply directly to formal schemes.

Grothendieck Duality continues with the construction of a *pseudofunctor* agreeing with the above right adjoint on the category of proper maps, and with the usual inverse-image functor on the category of étale maps (cf. [**De,** §3, pp. 303–318]). A first step in this construction is showing that the construction of the said right adjoint is "local on the source." Following Verdier, we can deduce this from flat base change (§8.3, page 88). But we do not yet have a pseudofunctor for, say, separated pseudo-finite-type maps of formal schemes (see §1.2.2, page 7), because at present we lack a compactification theorem for such maps, analogous to the well-known one of Nagata for separated finite-type maps of ordinary noetherian schemes (factoring them as proper $\circ$ open immersion).

The role played by quasi-coherent sheaves on ordinary schemes is taken over on formal schemes by limits of coherent sheaves (which are quasi-coherent, though the converse doesn't always hold), see Theorem 4.1, page 41. The most general duality theorems are stated for quasi-coherent *torsion* sheaves—sheaves with sections annihilated locally by an *open* ideal (see §6, beginning on page 58). The expected statements also continue to hold for coherent sheaves on formal schemes (see e.g., Theorem 8.4, page 89), the transition from torsion sheaves being effected via Proposition 6.2.1, a special case of Greenlees-May duality.

Greenlees-May duality on noetherian formal schemes (first proved in [**GM**] for modules over rings) is the subject of the second paper. This is a canonical duality between the right-derived torsion functor and the left-derived completion functor associated to a coherent ideal on a formal scheme—a far-reaching generalization of classical local duality.

Though noetherian hypotheses play an essential role in the formal context, they can be largely eliminated for ordinary schemes, and that is the subject of the third paper. An underlying strategy illustrated there is to reduce as much as possible of the complexity of duality theory, both in definitions of maps and in proofs of theorems, to a purely formal category-theoretic context. Thus we use resolutions (injective or flat) of complexes only to establish a few basic formal relations—the axioms, so to speak; and after that the deduction of commutativity of various functorial diagrams becomes a central issue, one which was unduly slighted in the standard treatments (see e.g., p. 117 in [**Ha**]).[1] The categorical context is generated by five of the famous "six operations" (derived tensor and Hom, inverse and direct image, and a right adjoint for the latter—compactly-supported direct images not being considered here). It is very rich in relations. Dealing directly with the variety of possible commutativities, at first amusing, then tedious, appears before long to outgrow human capacities. This sort of situation is treated in the literature under the rubric "coherence in categories"; but the tools developed in that area seem as yet unable to offer much in the present complex setup.

Could logicians or computer scientists help?

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[1] In [**Co**], his reworking and clarification of [**Ha**], Conrad does come to grips with this issue.

ISBN 0-8218-1942-9

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