CONTEMPORARY MATHEMATICS

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The Geometrical Study of Differential Equations

NFS–CBMS Conference on
The Geometrical Study of Differential Equations
June 20–25, 2000
Howard University
Washington, D.C.

Joshua A. Leslie
Thierry P. Robart
Editors

American Mathematical Society

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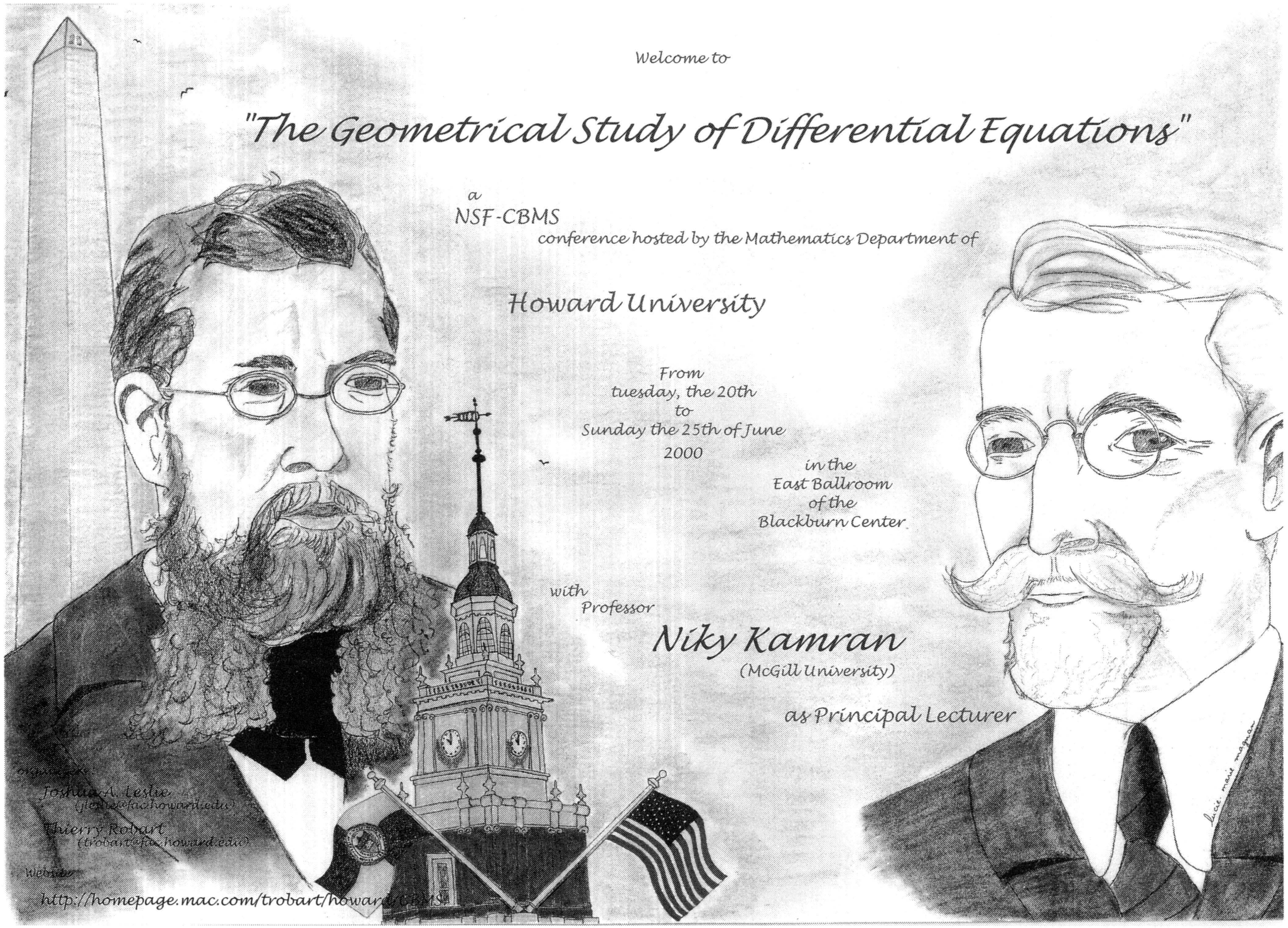

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This volume contains the proceedings of the NSF–CBMS Conference on "The Geometrical Study of Differential Equations", which was held at Howard University, Washington, D.C., June 20–25, 2000 with support from the Conference Board of the Mathematical Sciences and the National Science Foundation.

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Contents

Foreword

These are the proceedings of the NSF-CBMS conference on "The Geometrical Study of Differential Equations". The five day conference was held at Howard University in June 2000 under a sunny Washingtonian sky. It featured Niky Kamran (McGill University) as principal speaker.

This volume is published as a companion book to the CBMS monograph based on the principal lectures. The stress has been placed on ideas rather than technique. It contains papers presented (or that could have been presented) at the conference. The speakers were asked to give twenty minute non-technical talks bearing the diverse and non-specialist nature of their audience in mind. The CBMS conference, with this characteristic blend of ten principal lectures and short additional talks, turned out to be very successful. The talks have been substituted here by about ten page contributions. These collected papers, in presenting some important recent developments in this area, offer a rich source of inspiration for new or established researchers in the field. All articles have been refereed.

The *geometrical study of differential equations* for combining two fundamental concepts deeply rooted in sciences is certainly one of the most fascinating fields of research. Despite its already long and distinguished history it also stands, at the beginning of this new century, as one of the most promising. Today, the considerable number of recent important papers makes it difficult for any non-specialist to take part actively in this research. We hope that this volume associated with Niky Kamran's monograph will ease that task and serve as a useful catalyst.

List of authors

I. M. Anderson (Utah State University), Richard Beals (Yale University), Peter A. Clarkson (University of Kent at Canterbury), Mark E. Fels (Utah State University), Rafael Hernandez Heredero (Universidad Complutense de Madrid), Peter E. Hydon (University of Surrey), Vladimir Itskov (University of Minnesota), Thomas A. Ivey (College of Charleston), Irina A. Kogan (Yale University), Elizabeth L. Mansfield (University of Kent at Canterbury), Gloria Mari-Beffa (University of Wisconsin), Robert Milson (Dalhousie University), Clara Nucci (Universita' di Perugia), Michael Oberguggenberger (Universität Innsbruck), Peter J. Olver (University of Minnesota), Enrique G. Reyes (Yale University), Thierry Robart (Howard University), Jan A. Sanders (Vrije Universiteit), Barbara Shipman (University of Texas at Arlington), C. G. Torre (Utah State University), V. Torrisi (Universita' di Perugia), Jing Ping Wang (Vrije Universiteit).

Collected papers

We indicate to which principal lecture(s)[1] each contribution could be related.

- "An Overview of Lie's Line-Sphere Correspondence" by R. Milson (CBMS Lecture 2).
- "Application of Lie Group Analysis to a Mathematical Model which Describes HIV Transmission " by V. Torrisi & M. C. Nucci (CBMS Lecture 2).
- "Geometry and PDE on the Heisenberg group: a case study" by Richard Beals (CBMS Lectures 1, 2, 3).
- "Invariant Evolutions of Curves and Surfaces and Completely Integrable Hamiltonian Systems" by Gloria Mari-Beffa (CBMS Lectures 4, 5, 6).
- "On the Fixed Points of the Toda Hierarchy" by Barbara Shipman (CBMS Lecture 4).
- "Group Invariant Solutions in Mathematical Physics and Differential Geometry" by I. M. Anderson & M. E. Fels (CBMS Lectures 2, 3).
- "Discrete Symmetries of Differential Equations" by P. E. Hydon (CBMS Lectures 2, 3, 7).
- "Integrable Geometric Evolution Equations for Curves" by Thomas A. Ivey (CBMS Lectures 3, 4, 9).
- "On Integrability of Evolution Equations and Representation Theory" by Jan A. Sanders & Jing Ping Wang (CBMS Lectures 2, 6, 9).
- "Symmetry Groups, Nonlinear Partial Differential Equations, and Generalized Functions" by M. Oberguggenberger (CBMS Lectures 2, 3).
- "Lie Symmetries of Differential-Difference Equations " by R. Hernandez Heredero (CBMS Lectures 2, 3).
- "On a Variational Complex for Difference Equations" by Elizabeth L. Mansfield & Peter E. Hydon (CBMS Lecture 7).
- "The Invariant Variational Bicomplex" by Irina A. Kogan & Peter J. Olver (CBMS Lectures 7, 8).
- "On Geometrically Integrable Equations and Hierarchies of Pseudo-spherical Type" by Enrique G. Reyes (CBMS Lectures 4, 5, 9).
- "Inductive Construction of Moving Frames" by Irina A. Kogan (CBMS Lectures 7, 8, 10).
- "Orbit Reduction of Contact Ideals and Group-Invariant Variational Problems" by V. Itskov (CBMS Lectures 7, 8, 10).
- "About the Local and Formal Geometry of PDE" by T. Robart (CBMS Lectures 3, 10).
- "Open Problems in Symmetry Analysis" by Peter A. Clarkson & Elizabeth L. Mansfield (CBMS Lectures 2, 3).

Acknowledgment

We wish to thank the Conference Board of Mathematical Sciences and the National Science Foundation for having made this Regional Conference possible. It has been for us a great pleasure to organize the conference according to the

[1]N. Kamran *Lectures on the geometrical study of differential equations*, CBMS-NSF Regional Conf. Ser., AMS, 2001 or 2002

CBMS spirit. Our goal would be reached if this spirit could, at least partially, show through this publication.

We would also like to express our special thanks to the American Mathematical Society for accepting these proceedings as a Contemporary Mathematics volume.

Our warmest thanks go finally to all invited speakers/contributors/referees for respecting the rules with ease and talent and in particular to Peter Olver who has been supportive from the very beginning. Of course we won't forget Niky Kamran for his delightful lectures and natural enthusiasm.

We dedicate this volume to Adèle, born on the 20th of March 2001, and her mother Lucie-Marie Magnan who kindly designed our beautiful poster.

Joshua A. Leslie & Thierry P. Robart
June 29, 2001

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Summary of the Principal Lectures

The principal lectures[1] correspond to the spirit of the NSF-CBMS Regional Research Conferences Program by being aimed at both the new and recent entrants in the area, including post-doctoral fellows and graduate students, and the established researchers in the field. They should thus be accessible to a wide audience of mathematicians interested in the subject. We reproduce here Niky Kamran's summary of his lectures.

Lecture 1. Differential Equations and their Geometry. Our aim in this first lecture will be to motivate some of the main ideas and principles underlying the differential geometrical study of differential equations. These will form the thread unifying the whole series of lectures. We will first explain how one attaches a geometric structure and local invariants to a differential equation by viewing it geometrically as a locus in a jet space. We will then outline the concepts of symmetry, normal forms, conservation laws, variational principles and geometric complete integrability. These themes will be explained and illustrated with the help of a variety of simple yet enlightening examples coming from differential geometry and mathematical physics.

Lecture 2. External Symmetries of Differential Equations. External symmetries of differential equations were first discovered and studied by Lie. An external symmetry is a vector field in the ambient jet space which, when restricted to the locus defined by the differential equation, is tangent to that locus. The flow of a symmetry thus maps solutions to solutions. Lie proved that the existence of non-trivial external symmetries allows one in many cases to construct group-invariant solutions which are governed by a differential equation involving fewer independent variables. This principle of reduction by symmetry has thus become a very powerful and widely used tool for constructing exact solutions. It can be said to be one of the most successful applications of the geometric study of differential equations. We will present and illustrate by means of examples the main results concerning Lie symmetries, and discuss the particularly attractive form that they take in the Lagrangian and Hamiltonian contexts.

Lecture 3. Internal and Generalized Symmetries There are, besides the external symmetries studied in the preceding lecture, two additional classes of symmetries which are of great importance for the geometric study of differential equations. The first is the class of internal symmetries, which arise when differential

[1] N. Kamran *Lectures on the geometrical study of differential equations*, CBMS-NSF Regional Conf. Ser., AMS, 2001 or 2002

equations are viewed intrinsically as exterior differential systems. Internal symmetries were discovered by E. Cartan in his fundamental work on concrete realizations of Lie transformation groups by infinitesimal transformations. The second is the class of generalized symmetries, which are formal symmetry vector fields depending on derivatives of arbitrary order. They were discovered by E. Nöther in her famous theorems relating symmetries of variational problems to conservation laws. We will give a motivated introduction to these two types of symmetry, discuss their significance in terms of the integrability of the underlying differential equation and review what is known about the fascinating interplay between these three different kinds of symmetries. We will also outline some of the deceptively simple yet deep and open questions stemming from this study.

Lecture 4. Transformations of Surfaces and Differential Equations. Another important aspect of the relationship between partial differential equations and differential geometry stems from the fact that most of the local properties of submanifolds of Euclidean or projective spaces are expressible in terms of partial differential equations. This correspondence can be useful in two ways. From our knowledge of the geometry of submanifolds, we can obtain interesting solutions to these partial differential equations. Conversely, we can infer geometric properties of the manifolds and even prove the non-existence of certain geometric structures on manifolds from our knowledge of the solutions of the corresponding partial differential equations. This relationship has been particularly well studied in the case of surfaces, notably through the Bäcklund transformation of pseudo-spherical surfaces. Another well-known example of a geometric transformation of surfaces which has recently risen to prominence in the subject is the Laplace transformation of second-order linear hyperbolic equations in the plane. This transformation plays a significant role in the study of integrable systems. We will present the geometric and analytic formulations of the Laplace transformation of surfaces and review its most important connections with the theory of completely integrable systems. We will conclude by discussing some important open problems.

Lecture 5. Transformations of Submanifolds and Differential Equations. A major theme in the current research in the geometry of differential equations arises from the quest for a better understanding of the rarity of the complete integrability property for equations in more than two independent variables. There has therefore been a lot of interest in extending the known transformations of surfaces, such as the Bäcklund and Laplace transformations, to higher dimensions. In this lecture, we will present the multi-dimensional Laplace transformation that we have recently introduced for submanifolds of $\mathbf{P}^n$ with a conjugate coordinate net, and study its role in the exact solvability of certain second-order higher-dimensional systems. We will see in the next lecture that these systems are central to the theory of Hamiltonian systems of hydrodynamic type, which are rich in conservation laws.

Lecture 6. Applications to Hamiltonian Systems of Hydrodynamic Type. The class of Hamiltonian systems of hydrodynamic type, provides an interesting arena of applications for many of the concepts introduced above. We will see that the overdetermined system of linear partial differential equations that governs the conserved densities for such a system is precisely of the type to which the higher dimensional Laplace transformation is applicable. As a result, we will have a geometric method for constructing families of integrable systems of hydrodynamic type which are rich in conservation laws, and whose commuting flows can be determined explicitly. If time permits, we will mention some of the connections between this class of hydrodynamic systems and the theory of orthogonal coordinate systems in Riemannian geometry, as well as Lie sphere geometry.

Lecture 7. The Variational Bicomplex. Our aim in this lecture will be to introduce and review the main properties of the variational bi-complex. This is a bigraded complex of differential forms which generalizes the de Rham complex to differential forms on jet spaces. It provides a natural geometric framework for the study of variational principles, which will be the topic of Lecture 8, and conservation laws, which we will investigate in Lecture 9. The variational bicomplex also has an equivariant counterpart which plays an important role in the study of characteristic classes. We will introduce the calculus of differential forms in the variational bi-complex, and present the basic local and global exactness results which will be needed in Lecture 8.

Lecture 8. Applications to the Inverse Problem of the Calculus of Variations. One of the fundamental questions that one can ask about a differential operator is whether it can be expressed as the Euler-Lagrange operator of a variational problem. This is known as the inverse problem of the calculus of variations. We will start by reviewing the local formulation of this problem on Euclidean spaces and its global formulation on manifolds. We will then present the solution of both the local and global inverse problems of the calculus of variations in the geometric context of the variational bi-complex. We will notably see how the cohomology of the edge complex of the variational bi-complex gives rise to the obstructions to the local and global inverse problems. We will end our lecture with some open problems related to the equivariant inverse problem.

Lecture 9. Darboux Integrability and Conservation Laws. The most general and powerful notion of geometric integrability for second-order hyperbolic partial differential equations in the plane is that of Darboux integrability, which gives rise to solutions depending on two arbitrary functions of one variable and finitely many derivatives. It is thus a broad generalization to non-linear equations of d'Alembert's solution of the wave equation in one space variable. In geometric terms, Darboux integrability can be expressed by the presence of two foliations whose leaves constrain the characteristic curves of the equation. The simplest example of a nonlinear hyperbolic equation which is Darboux-integrable is given by Liouville's equation $u_{xy} = exp\, u$. It is a non-trivial problem to determine a workable set of necessary and sufficient conditions for a partial differential equation to be Darboux integrable at any order. We will review some recent work in which this problem is solved from the point of view of the constrained variational bi-complex. Darboux integrable equations will thus be characterized purely in terms of the behavior of their geometric invariants. We will also show how to compute conservation laws in this geometric context. We will conclude by presenting some important open problems concerning the analytic interpretation of the higher-degree conservation laws.

Lecture 10. An Introduction to the Characteristic Cohomology of Exterior Differential Systems. In this last lecture of the series, we will give an introduction to one of the most important recent advances in the study of the geometry of differential equations, namely the characteristic cohomology theory of exterior differential systems. This concept has been defined and studied in depth by Bryant an Griffiths. The characteristic cohomology theory provides a very general framework for the calculation and study of conservation laws, transformations (both local and non-local) and normal forms for exterior differential systems corresponding to differential equations. It also helps to explain the geometric mechanisms which underlie the existence of multi-valued solutions of differential equations.

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